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ORIGINAL ARTICLE

Assessment of refined higher order theories for the static and vibration analysis of laminated composite cylindrical shells

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ABSTRACT – In the present study, a generalized shell theory is presented and applied for the analysis of laminated composite cylindrical shells. A theoretical unification of the several refined shell theories is presented. The principle of work done is employed to derive five differential equations corresponding to five unknowns involved in the present generalized shell theory. Five differential equations are solved by an analytical procedure suggested by the Navier. The numerical results for simply supported laminated composite cylindrical shells are presented and compared with 3D elasticity solutions. Displacements, stresses and fundamental frequencies are obtained for isotropic, orthotropic, $0^0/90^0$ and $0^0/90^0/0^0$ laminated cylindrical shells. The numerical results are obtained for h/a=0.1, a/b=1 and different values of R/a ratio. Displacements and stresses of laminated cylindrical shells are estimated under sinusoidal transverse load. In the case of free vibration analysis, first five natural frequencies are presented. It is observed that refined theories predicts displacements and stresses in close agreement with 3D elasticity solutions whereas the FST and the CST underpredict the displacements and stresses. It is also observed that the CST overestimates the natural frequencies due to neglect of shear deformation effect.

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INTRODUCTION

Laminated composite cylindrical shells are widely used in many engineering industries due to their attractive structural properties such as high strength-to-weight and stiffness-to-weight ratios. Among equivalent single layer theories, layerwise theories and zig-zag theories; equivalent single layer theories are more popularly used for the analysis of laminated composite shells due to simplicity of mathematics. The classical shell theory (CST) of Kirchhoff [1] and first order shear deformation theory (FST) of Mindlin [2] have been widely used theories for the analysis of laminated composite thin shells. However, both the theories have certain drawbacks and not capable enough to predict accurate static and dynamic behaviour of thick laminated shells. This forces researchers to formulate refined models which accurately predict the global response of the laminated shells. In these higher order shell theories, the displacements are expanded using polynomial or non-polynomial type strain functions. In the open literature, Bhimaraddi [3] and Reddy [4] have presented polynomial type shear deformation theories whereas Levy [5], Soldatos [6] and Karama et al. [7] have developed nonpolynomial type shear deformation theories such as trigonometric, hyperbolic and exponential shear deformation theories, respectively. Aydogdu [8] has developed a new shear deformation theory for the bending, buckling and free vibration analysis of laminated composite plates which is considered as an improvement over exponential shear deformation theory. Akavci [9] has developed two hyperbolic models for the analysis laminated composite plates. Mantari et al. [10] have developed higher order shear deformation theory considering trigonometric and exponentail functions for the analysis of laminated composite plates and shells. Neves et al. [11, 12], have developed sinusoidal and hyperbolic shear deformation theories considering the effects of transverse normal strain for the analysis of laminated composite plates. Ghugal and Sayyad [13], and Sayyad and Ghugal [14] have developed a trigonometric shear deformation theory for the bending, buckling and free vibration analysis of laminated composite and sandwich plates. Recently, Sayyad and Naik [15] have developed a fifth order shear deformation theory considering the effects of transverse shear and normal deformation for the interlaminar stress analysis of laminated plates. Detailed discussions on these theories have been presented by Sayyad and Ghugal [16], Liew et al. [17] and Qatu et al. [18].

A literature on static and free vibration analysis of cylindrical shells using various classical and higher order shell theories is published by many researchers. Reddy [19] have applied the third order shear deformation theory for the analysis of laminated composite shells. Soldatos and Timarci [20] and; Timarci and Soldatos [21] have presented an unified formulation of higher order shell theories for the free vibration analysis of laminated omposite cylindrical shells. Zenkour and Fares [22] have developed a refined first order theory for the thermal bending analysis of laminated composite cylindrical shells. Khdeir [23] have presented static and vibration analysis of cross-ply shells using thick shell theory Asadi et al. [24] have presented three-dimensional solutions along with application of various shear deformation theories for the static and vibration analyses of laminated composite cylindrical shells. Khalili et al. [25]] have presented free vibration analysis of homogeneous isotropic circular cylindrical shells based on a three-dimensional refined higher-order theory. Carrera and Brischetto [26] and Carrera et al. [26, 27] have presented analysis of laminated shells using

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various shell theories recovered from the Carrera's unified formulation. Mantari et al. [28] have developed a new higher order shear deformation theory for the static and free vibration analysis of laminated composite shells. However, from the aforementioned literature it is found that most of the researchers have presented only transverse deflection quantities during the static analysis of laminated shells. Values of in-plane normal stresses and transverse shear stresses for laminated composite shells under mechanical/thermal loads are not reported by many researchers in their papers. It is well known that equivalent single layer shell theories fail to satisfy the continuity at the layer interface in case of laminated shells. Therefore, transverse shear stresses are recovered by using equilibrium equations of theory of elasticity to ascertain continuity at the layer interface. Detail procedure of this method is given by Tornabene et al. [29–35] and; Sayyad and Ghugal [36].

In this work, a generalized displacement model is presented to recover several equivalent single layer higher order and classical shell theories such as the classical shell theory (CST) of Kirchhoff [1], first order shell theory (FST) of Mindlin [2], parabolic shell theory (PST) of Reddy [4], trigonometric shell theory (TST) of Levy [5], hyperbolic shell theory (HST) of Soldatos [6] and exponential shell theory (EST) of Karama et al. [7]. Further, these theories are applied for the static and free vibration analysis of laminated composite cylindrical shells. Five differential equations of the present generalized displacement model are derived using the principle of virtual work. Solutions for static and free vibration problems of simply supported cylindrical shells are obtained using the Navier's technique. A computer code is developed in Fortran 77 to determine displacements, stresses and frequencies. Numerical results are compared with three-dimensional elasticity solutions given by Bhimaraddi and Chandrashekhara [37] for static analysis. Bhimaraddi [38] has also provided three-dimensional elasticity solutions for doubly curved laminated shells. Since, 3D elasticity solution for the free vibration analysis is not available in the literature; the present results are compared with other works available in the literature. In this work, transverse shear stresses are recovered from 3D stress equilibrium equations of elasticity to ascertain continuity at layer interface/s of the laminated cylindrical shells.

MATHEMATICAL FORMULATION

A laminated composite cylindrical shell of rectangular planform ($a\times b$) and thickness h shown in Figure 1 is considered for the mathematical formulation. $\boldsymbol{R}$ denotes the principal radius of curvature of the middle surface. Cylindrical shell is composed of a $\boldsymbol{N}$ number of orthotropic layers perfectly bonded together.

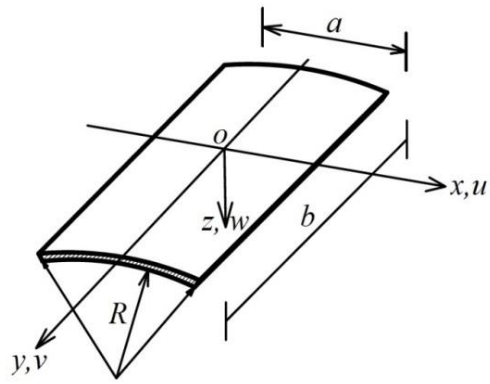

Figure 1. Laminated cylindrical shell under consideration

In the present generalized displacement model, in-plane displacements are presented in three components (extension, bending and shear) whereas the transverse displacement is assumed to be function of $\boldsymbol{x}$ and $\boldsymbol{y}$ coordinates. In the Eq. (1) u, v and w represent the displacements of any point of the shell domain whereas u_0, v_0, w_0 represent the displacements of any point on the middle surface of the shell in the x-, y- and z-directions, respectively; $\zeta(z)$ represent the shape functions used to recover various refined higher order and classical shell theories from the present generalized displacement model.

$$u(x,y,z,t) = \left(1+\frac{z}{R}\right)u_0(x,y,t) - z\frac{\partial w_0}{\partial x} + \zeta(z)\phi(x,y,t) \tag{1a}$$

$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y} + \zeta(z)\psi(x,y,t) \tag{1b}$$

$$w(x,y,t) = w_0(x,y,t) \tag{1c}$$

Following are the strain components obtained using Eqs. (1a)-(1c) and the strain–displacement relations from the theory of elasticity [39].

$$\varepsilon_x = \varepsilon_x^0 + z\varepsilon_x^1 + \zeta(z)\varepsilon_x^2 \tag{2a}$$

$$\varepsilon_y = \varepsilon_y^0 + z\varepsilon_y^1 + \zeta(z)\varepsilon_y^2 \tag{2b}$$

$$\gamma_{xy} = \gamma_{xy}^0 + z\gamma_{xy}^1 + \zeta(z)\gamma_{xy}^2 \tag{2c}$$

$$\gamma_{xz} = \zeta'(z)\gamma_{xz}^0 \tag{2d}$$

$$\gamma_{yz} = \zeta'(z)\gamma_{yz}^0 \tag{2e}$$

Where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x} + \frac{w_0}{R},\quad \varepsilon_x^1 = -\frac{\partial^2 w_0}{\partial x^2},\quad \varepsilon_x^2 = \frac{\partial \phi}{\partial x},\quad \varepsilon_y^0 = \frac{\partial v_0}{\partial y},\quad \varepsilon_y^1 = -\frac{\partial^2 w_0}{\partial y^2},\quad \varepsilon_y^2 = \frac{\partial \psi}{\partial y}, \tag{3a}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x},\quad \gamma_{xy}^1 = -2\frac{\partial^2 w_0}{\partial x \partial y},\quad \gamma_{xy}^2 = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x},\quad \gamma_{xz}^0 = \phi,\quad \gamma_{yz}^0 = \psi \tag{3b}$$

In the Eqs. (2a)-(2e), prime indicates derivative with respect to *z*. The stresses in the k^{th} layer of the laminated cylindrical shells are obtained using the following constitutive relationship [39].

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)} \tag{4}$$

where Q_{ij} are the stiffness coefficients; *x, y, z* represent laminate axes and 1, 2, 3 represent material axes. Stiffness coefficients are expressed in-terms of engineering constants as follows [39].

$$Q_{11} = \frac{E_1}{1-\mu_{12}\mu_{21}},\quad Q_{12} = \frac{\mu_{21}E_1}{1-\mu_{12}\mu_{21}},\quad Q_{22} = \frac{E_2}{1-\mu_{12}\mu_{21}},\quad Q_{66} = G_{12},\quad Q_{55} = G_{13},\quad Q_{44} = G_{23} \tag{5}$$

The principle of virtual work is employed to formulate five differential equations associated with the present generalized displacement model. Following is the analytical form of the principle of virtual work where δ denotes the virtual operator.

$$\begin{aligned} &\int_V \left(\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{zx}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz} + \tau_{xy}\delta\gamma_{xy}\right)dV - \int_A q(x,y)\delta w\, dA \\ &-\rho\int_V \left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w}{\partial t^2}\delta w\right)dV = 0 \end{aligned} \tag{6}$$

By putting Eqs. (2a)-(2e) into the Eq. (6), one can get,

$$\begin{aligned} &\int_{dV} \Big\{ \sigma_x\left[\varepsilon_x^0 + z\varepsilon_x^1 + \zeta(z)\varepsilon_x^2\right] + \sigma_y\left[\varepsilon_y^0 + z\varepsilon_y^1 + \zeta(z)\varepsilon_y^2\right] + \tau_{xy}\left[\gamma_{xy}^0 + z\gamma_{xy}^1 + \zeta(z)\gamma_{xy}^2\right] \\ &+\tau_{xz}\zeta'(z)\gamma_{xz}^0 + \tau_{yz}\zeta'(z)\gamma_{yz}^0 \Big\} dV - \int_A q(x,y)\delta w\, dA + \rho\int_V \left(\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w}{\partial t^2}\delta w\right)dV = 0 \end{aligned} \tag{7}$$

The force and moment resultants can be introduced after performing integrations with respect to ***z*** coordinate where superscript ***b*** is used for the resultants due to bending whereas superscript ***s*** is used for the resultants due to shear [39].

$$\left\{N_x \quad M_x^b \quad M_x^s\right\}^T = \sum_{k=1}^{N} \int_{h_k}^{h_{k+1}} \left\{1 \quad z \quad \zeta(z)\right\}^T \sigma_x^k \, dz \tag{8a}$$

$$\left\{N_y \quad M_y^b \quad M_y^s\right\}^T = \sum_{k=1}^{N} \int_{h_k}^{h_{k+1}} \left\{1 \quad z \quad \zeta(z)\right\}^T \sigma_y^k \, dz \tag{8b}$$

$$\left\{N_{xy} \quad M_{xy}^b \quad M_{xy}^s\right\}^T = \sum_{k=1}^{N} \int_{h_k}^{h_{k+1}} \left\{1 \quad z \quad \zeta(z)\right\}^T \tau_{xy}^k \, dz \tag{8c}$$

$$\left\{Q_x \quad Q_y\right\}^T = \sum_{k=1}^{N} \int_{h_k}^{h_{k+1}} \left\{\tau_{xz} \quad \tau_{yz}\right\}^T \zeta'(z) \, dz \tag{8d}$$

Force and moment resultants of Eqs. (8a)-(8d) can be expressed in terms of unknown variables as follows.

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x^b \\ M_y^b \\ M_{xy}^b \\ M_x^s \\ M_y^s \\ M_{xy}^s \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{11} & As_{11} & A_{12} & B_{12} & As_{12} & 0 & 0 & 0 \\ A_{12} & B_{12} & As_{12} & A_{22} & B_{22} & As_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{66} & B_{66} & As_{66} \\ B_{11} & D_{11} & Bs_{11} & B_{12} & D_{12} & Bs_{12} & 0 & 0 & 0 \\ B_{12} & D_{12} & Bs_{12} & B_{22} & D_{22} & Bs_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{66} & D_{66} & Bs_{66} \\ As_{11} & Bs_{11} & Ass_{11} & As_{12} & Bs_{12} & Ass_{12} & 0 & 0 & 0 \\ As_{12} & Bs_{12} & Ass_{12} & As_{22} & Bs_{22} & Ass_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & As_{66} & Bs_{66} & Ass_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_x^1 \\ \varepsilon_x^2 \\ \varepsilon_y^0 \\ \varepsilon_y^1 \\ \varepsilon_y^2 \\ \gamma_{xy}^0 \\ \gamma_{xy}^1 \\ \gamma_{xy}^2 \end{Bmatrix} \tag{9}$$

and

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} Acc_{55} & 0 \\ 0 & Acc_{44} \end{bmatrix} \begin{Bmatrix} \phi \\ \psi \end{Bmatrix} \tag{10}$$

Using Eqs. (8a)-(8d), one can modify Eq. (7) as follows

$$\begin{aligned} &\int_0^b \int_0^a \Big\{ N_x \delta\varepsilon_x^0 + M_x^b \delta\varepsilon_x^1 + M_x^s \delta\varepsilon_x^2 + N_y \delta\varepsilon_y^0 + M_y^b \delta\varepsilon_y^1 + M_y^s \delta\varepsilon_y^2 + N_{xy} \delta\gamma_{xy}^0 + M_{xy}^b \delta\gamma_{xy}^1 + M_{xy}^s \delta\gamma_{xy}^2 + \\ &Q_x \delta\gamma_{xz}^0 + Q_y \delta\gamma_{yz}^0 - q\,\delta w_0 \Big\} dx\,dy + \int_0^b \int_0^a \Big\{ + \left(I_1 + 2\frac{I_2}{R} + \frac{I_3}{R^2}\right) \frac{\partial^2 u_0}{\partial t^2} \delta u_0 - \left(I_2 + \frac{I_3}{R}\right) \frac{\partial^3 w_0}{\partial x \partial t^2} \delta u_0 \\ &+ \left(I_4 + \frac{I_5}{R}\right) \frac{\partial^2 \phi}{\partial t^2} \delta u_0 - \left(I_2 + \frac{I_3}{R}\right) \frac{\partial^2 u_0}{\partial t^2} \frac{\partial \delta w_0}{\partial x} + I_3 \frac{\partial^3 w_0}{\partial x \partial t^2} \frac{\partial \delta w_0}{\partial x} - I_5 \frac{\partial^2 \phi}{\partial t^2} \frac{\partial \delta w_0}{\partial x} + \left(I_4 + \frac{I_5}{R}\right) \frac{\partial^2 u_0}{\partial t^2} \delta\phi - \\ &I_5 \frac{\partial^3 w_0}{\partial x \partial t^2} \delta\phi + I_6 \frac{\partial^2 \phi}{\partial t^2} \delta\phi + I_1 \frac{\partial^2 v_0}{\partial t^2} \delta v_0 - I_2 \frac{\partial^3 w_0}{\partial y \partial t^2} \delta v_0 + I_4 \frac{\partial^2 \psi}{\partial t^2} \delta v_0 - I_2 \frac{\partial^2 v_0}{\partial t^2} \frac{\partial \delta w_0}{\partial y} + I_3 \frac{\partial^3 w_0}{\partial y \partial t^2} \frac{\partial \delta w_0}{\partial y} - \\ &I_5 \frac{\partial^2 \psi}{\partial t^2} \frac{\partial \delta w_0}{\partial y} + I_4 \frac{\partial^2 v_0}{\partial t^2} \delta\psi - I_5 \frac{\partial^3 w_0}{\partial y \partial t^2} \delta\psi + I_6 \frac{\partial^2 \psi}{\partial t^2} \delta\psi + I_1 \frac{\partial^2 w_0}{\partial t^2} \Big\} dx\,dy = 0 \end{aligned} \tag{11}$$

After performing integration of the Eq. (11) by parts, collecting the terms of $\delta u_0, \delta v_0, \delta w_0, \delta\phi$ and $\delta\psi$ and setting them equal to zero; one can write the following differential equations associated with the present generalized displacement model.

$$\delta u_0 : \quad \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \left(I_1 + 2\frac{I_2}{R} + \frac{I_3}{R^2}\right) \frac{\partial^2 u_0}{\partial t^2} - \left(I_2 + \frac{I_3}{R}\right) \frac{\partial^3 w_0}{\partial x \partial t^2} + \left(I_4 + \frac{I_5}{R}\right) \frac{\partial^2 \phi}{\partial t^2} \tag{12a}$$

$$\delta v_0 : \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_1 \frac{\partial^2 v_0}{\partial t^2} - I_2 \frac{\partial^3 w_0}{\partial y \partial t^2} + I_4 \frac{\partial^2 \psi}{\partial t^2} \tag{12b}$$

$$\delta w_0: \quad \frac{\partial^2 M_x^b}{\partial x^2}+2\frac{\partial^2 M_{xy}^b}{\partial x \partial y}+\frac{\partial^2 M_y^b}{\partial y}-\frac{N_x}{R}+q=\left(I_2+\frac{I_3}{R}\right)\frac{\partial^3 u_0}{\partial x \partial t^2}-I_3\frac{\partial^4 w_0}{\partial x^2 \partial t^2}$$
$$+I_5\frac{\partial^3 \phi}{\partial x \partial t^2}+I_2\frac{\partial^3 v_0}{\partial y \partial t^2}-I_3\frac{\partial^4 w_0}{\partial y^2 \partial t^2}+I_5\frac{\partial^3 \psi}{\partial y \partial t^2}+I_1\frac{\partial^2 w_0}{\partial t^2} \tag{12c}$$

$$\delta \phi: \quad \frac{\partial M_x^s}{\partial x}+\frac{\partial M_{xy}^s}{\partial y}-Q_x=\left(I_4+\frac{I_5}{R}\right)\frac{\partial^2 u_0}{\partial t^2}-I_5\frac{\partial^3 w_0}{\partial x \partial t^2}+I_6\frac{\partial^2 \phi}{\partial t^2} \tag{12d}$$

$$\delta \psi: \quad \frac{\partial M_y^s}{\partial y}+\frac{\partial M_{xy}^s}{\partial x}-Q_y=I_4\frac{\partial^2 v_0}{\partial t^2}-I_5\frac{\partial^3 w_0}{\partial y \partial t^2}+I_6\frac{\partial^2 \psi}{\partial t^2} \tag{12e}$$

where

$$\{A_{ij} \quad B_{ij} \quad D_{ij}\}=\sum_{k=1}^{N} Q_{ij}^k \int_{h_k}^{h_{k+1}} \{1 \quad z \quad z^2\}dz, \qquad (i,j=1,2,3,6) \tag{13a}$$

$$\{As_{ij} \quad Bs_{ij} \quad Ass_{ij}\}=\sum_{k=1}^{N} Q_{ij}^k \int_{h_k}^{h_{k+1}} \zeta(z)\{1 \quad z \quad \zeta(z)\}dz, \quad (i,j=1,2,3,6) \tag{13b}$$

$$Acc_{ij}=\sum_{k=1}^{N} Q_{ij}^k \int_{h_k}^{h_{k+1}} [\zeta'(z)]^2 dz, \qquad (i,j=4,5) \tag{13c}$$

$$\{I_1 \quad I_2 \quad I_3\}=\sum_{k=1}^{N} \rho^k \int_{h_k}^{h_{k+1}} \{1 \quad z \quad z^2\}dz, \tag{13d}$$

$$\{I_4 \quad I_5 \quad I_6\}=\sum_{k=1}^{N} \rho^k \int_{h_k}^{h_{k+1}} \zeta(z)\{1 \quad z \quad \zeta(z)\}dz \tag{13e}$$

The boundary conditions along the four edges of the shell are presented in Table 1.

Table 1. The boundary conditions along the four edges of the shell

along x = 0 and x = a	along y = 0 and y = b
$N_x=0$ or $u_0=0$	$N_{xy}=0$ or $u_0=0$
$N_{xy}=0$ or $v_0=0$	$N_y=0$ or $v_0=0$
$V_x=0$ or $w=0$	$V_y=0$ or $w=0$
$M_x^b=0$ or $\frac{\partial w}{\partial x}=0$	$M_y^b=0$ or $\frac{\partial w}{\partial y}=0$
$M_x^s=0$ or $\phi=0$	$M_{xy}^s=0$ or $\phi=0$
$M_{xy}^s=0$ or $\psi=0$	$M_y^s=0$ or $\psi=0$

where

$$V_x=\partial M_x^b/\partial x+2\partial M_{xy}^b/\partial y, \quad V_y=\partial M_y^b/\partial y+2\partial M_{xy}^b/\partial x \tag{14}$$

Above mentioned five differential equations can be also written as follows in terms of unknown variables in the generalized displacement model

$$\delta u_0: \quad A_{11}\frac{\partial^2 u_0}{\partial x^2}+A_{66}\frac{\partial^2 u_0}{\partial y^2}+(A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x \partial y}+\frac{A_{11}}{R}\frac{\partial w_0}{\partial x}-B_{11}\frac{\partial^3 w_0}{\partial x^3}-(B_{12}+2B_{66})\frac{\partial^3 w_0}{\partial x \partial y^2}+As_{11}\frac{\partial^2 \phi}{\partial x^2}$$
$$+As_{66}\frac{\partial^2 \phi}{\partial y^2}+(As_{12}+As_{66})\frac{\partial^2 \psi}{\partial x \partial y}-\left(I_1+2\frac{I_2}{R}+\frac{I_3}{R^2}\right)\frac{\partial^2 u_0}{\partial t^2}+\left(I_2+\frac{I_3}{R}\right)\frac{\partial^3 w_0}{\partial x \partial t^2}-\left(I_4+\frac{I_5}{R}\right)\frac{\partial^2 \phi}{\partial t^2}=0 \tag{15}$$

$$\begin{aligned}\delta v_0:\quad &\left(A_{12}+A_{66}\right)\frac{\partial^2 u_0}{\partial x\partial y}+A_{22}\frac{\partial^2 v_0}{\partial y^2}+A_{66}\frac{\partial^2 v_0}{\partial x^2}-B_{22}\frac{\partial^3 w_0}{\partial y^3}-\left(B_{12}+2B_{66}\right)\frac{\partial^3 w_0}{\partial x^2\partial y}+\frac{A_{12}}{R}\frac{\partial w_0}{\partial y}\\ &+\left(As_{12}+As_{66}\right)\frac{\partial^2\phi}{\partial x\partial y}+As_{22}\frac{\partial^2\psi}{\partial y^2}+As_{66}\frac{\partial^2\psi}{\partial x^2}-I_1\frac{\partial^2 v_0}{\partial t^2}+I_2\frac{\partial^3 w_0}{\partial y\partial t^2}-I_4\frac{\partial^2\psi}{\partial t^2}=0\end{aligned}\tag{16}$$

$$\begin{aligned}\delta w_0:\quad &B_{11}\frac{\partial^3 u_0}{\partial x^3}+\left(B_{12}+2B_{66}\right)\frac{\partial^3 u_0}{\partial x\partial y^2}-\frac{A_{11}}{R}\frac{\partial u_0}{\partial x}+B_{22}\frac{\partial^3 v_0}{\partial y^3}+\left(B_{12}+2B_{66}\right)\frac{\partial^3 v_0}{\partial x^2\partial y}-\frac{A_{12}}{R}\frac{\partial v_0}{\partial y}+\frac{2B_{11}}{R}\frac{\partial^2 w_0}{\partial x^2}\\ &-D_{11}\frac{\partial^4 w_0}{\partial x^4}-2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_0}{\partial x^2\partial y^2}-D_{22}\frac{\partial^4 w_0}{\partial y^4}+\frac{2B_{12}}{R}\frac{\partial^2 w_0}{\partial y^2}-\frac{A_{11}}{R^2}w_0+Bs_{11}\frac{\partial^3\phi}{\partial x^3}\\ &+\left(Bs_{12}+2Bs_{66}\right)\frac{\partial^3\phi}{\partial x\partial y^2}-\frac{As_{11}}{R}\frac{\partial\phi}{\partial x}+Bs_{22}\frac{\partial^3\psi}{\partial y^3}+\left(Bs_{12}+2Bs_{66}\right)\frac{\partial^3\psi}{\partial x^2\partial y}-\frac{As_{12}}{R}\frac{\partial\psi}{\partial y}-q\\ &-\left(I_2+\frac{I_3}{R}\right)\frac{\partial^3 u_0}{\partial x\partial t^2}+I_3\frac{\partial^4 w_0}{\partial x^2\partial t^2}-I_5\frac{\partial^3\phi}{\partial x\partial t^2}-I_2\frac{\partial^3 v_0}{\partial y\partial t^2}+I_3\frac{\partial^4 w_0}{\partial y^2\partial t^2}-I_5\frac{\partial^3\psi}{\partial y\partial t^2}-I_1\frac{\partial^2 w_0}{\partial t^2}=0\end{aligned}\tag{17}$$

$$\begin{aligned}\delta\phi:\quad &As_{11}\frac{\partial^2 u_0}{\partial x^2}+As_{66}\frac{\partial^2 u_0}{\partial y^2}+\left(As_{12}+As_{66}\right)\frac{\partial^2 v_0}{\partial x\partial y}-Bs_{11}\frac{\partial^3 w_0}{\partial x^3}-\left(Bs_{12}+2Bs_{66}\right)\frac{\partial^3 w_0}{\partial x\partial y^2}+\frac{As_{11}}{R}\frac{\partial w_0}{\partial x}\\ &+Ass_{11}\frac{\partial^2\phi}{\partial x^2}+Ass_{66}\frac{\partial^2\phi}{\partial y^2}-Acc_{55}\phi+\left(Ass_{12}+Ass_{66}\right)\frac{\partial^2\psi}{\partial x\partial y}-\left(I_4+\frac{I_5}{R}\right)\frac{\partial^2 u_0}{\partial t^2}+I_5\frac{\partial^3 w_0}{\partial x\partial t^2}-I_6\frac{\partial^2\phi}{\partial t^2}=0\end{aligned}\tag{18}$$

$$\begin{aligned}\delta\psi:\quad &\left(As_{12}+As_{66}\right)\frac{\partial^2 u_0}{\partial x\partial y}+As_{66}\frac{\partial^2 v_0}{\partial x^2}+As_{22}\frac{\partial^2 v_0}{\partial y^2}-Bs_{22}\frac{\partial^3 w_0}{\partial y^3}-\left(Bs_{12}+2Bs_{66}\right)\frac{\partial^3 w_0}{\partial x^2\partial y}+\frac{As_{12}}{R}\frac{\partial w_0}{\partial y}\\ &+\left(Ass_{12}+Ass_{66}\right)\frac{\partial^2\phi}{\partial x\partial y}+Ass_{22}\frac{\partial^2\psi}{\partial y^2}+Ass_{66}\frac{\partial^2\psi}{\partial x^2}-Acc_{55}\psi-I_4\frac{\partial^2 v_0}{\partial t^2}+I_5\frac{\partial^3 w_0}{\partial y\partial t^2}-I_6\frac{\partial^2\psi}{\partial t^2}=0\end{aligned}\tag{19}$$

SOLUTIONS FOR SIMPLY-SUPPORTED SHELLS

In this section, solution procedure of the five differential equations is explained when those are applied for the simply supported laminated cylindrical shells. Navier's technique is the most widely adopted technique for the analysis of simply supported shells [39]. In this technique, unknowns are presented in the trigonometric form to satisfy the following boundary conditions of the simply supported edges.

$$\text{at edges } x=0 \text{ and } x=a:\ N_x=v_0=w=\psi=M_x^b=M_x^s=0 \tag{20a}$$

$$\text{at edges } y=0 \text{ and } y=b:\ N_y=u_0=w=\phi=M_y^b=M_y^s=0 \tag{20b}$$

The unknowns and transverse load are presented in the following trigonometric form.

For static problem

$$\{u_0 \quad \phi\}=\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\{u_{mn} \quad \phi_{mn}\}\cos(\alpha_1 x)\sin(\alpha_2 y) \tag{21a}$$

$$\{v_0 \quad \psi\}=\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\{v_{mn} \quad \psi_{mn}\}\sin(\alpha_1 x)\cos(\alpha_2 y) \tag{21b}$$

$$\{w_0 \quad q\}=\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\{w_{mn} \quad q_{mn}\}\sin(\alpha_1 x)\sin(\alpha_2 y) \tag{21c}$$

For free vibration problem (q=0)

$$\{u_0 \quad \phi\}=\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\{u_{mn} \quad \phi_{mn}\}\cos(\alpha_1 x)\sin(\alpha_2 y)e^{i\omega t} \tag{22a}$$

$$\{v_0 \quad \psi\} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \{v_{mn} \quad \psi_{mn}\} \sin(\alpha_1 x)\cos(\alpha_2 y) e^{i\omega t} \tag{22b}$$

$$\{w_0\} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \{w_{mn}\} \sin(\alpha_1 x)\sin(\alpha_2 y) e^{i\omega t} \tag{22c}$$

where, $\alpha_1 = m\pi / a$, $\alpha_2 = n\pi / b$; q_{mn} represents the Fourier coefficient of the transverse load, (m, n) are odd integers i.e. m=n=1,3,5…∞; in the case of sinusoidal transverse load, $q_{mn} = 1$; ω represents the natural frequency, t represents time, $i = \sqrt{-1}$ and $u_{mn}, v_{mn}, w_{mn}, \phi_{mn}, \psi_{mn}$ are the unknown coefficients to be determine. Substitution of Eqs. (21a)-(21c) and Eqs. (22a)-(22c) into the differential Eqs. (15)-(19) leads to the following systems of equations for static and free vibration problems respectively.

$$[K]\{\Delta\} = \{F\} \tag{23}$$

$$\{[K] - \omega^2 [M]\}\{\Delta\} = \{0\} \tag{24}$$

Displacements and stresses in homogenous and laminated cylindrical shells can be determined using solution of Eq. (23) whereas natural frequencies can be determined using solution of Eq. (24). The elements of stiffness matrix [K], force vector {F}, mass matrix [M] and vector of unknowns{ Δ }are given below.
Elements of stiffness matrix [K]

$$\begin{gathered}
K_{11} = A_{11}\alpha_1^2 + A_{66}\alpha_2^2, \quad K_{12} = (A_{12} + A_{66})\alpha_1\alpha_2, \quad K_{13} = -\left[\frac{A_{11}}{R}\alpha_1 + B_{11}\alpha_1^3 - (B_{12} + 2B_{66})\alpha_1\alpha_2^2\right], \\
K_{14} = As_{11}\alpha_1^2 + As_{66}\alpha_2^2, \quad K_{15} = (As_{12} + As_{66})\alpha_1\alpha_2, \quad K_{22} = A_{66}\alpha_1^2 + A_{22}\alpha_2^2, \\
K_{23} = -\left[\frac{A_{12}}{R}\alpha_2 + B_{22}\alpha_2^3 - (B_{12} + 2B_{66})\alpha_1^2\alpha_2\right], K_{24} = (As_{12} + As_{66})\alpha_1\alpha_2, K_{25} = As_{66}\alpha_1^2 + As_{22}\alpha_2^2, \\
K_{33} = D_{11}\alpha_1^4 + 2(D_{12} + 2D_{66})\alpha_1^2\alpha_2^2 + D_{22}\alpha_2^4 + \frac{2B_{11}}{R}\alpha_1^2 + \frac{2B_{12}}{R}\alpha_2^2 + \frac{A_{11}}{R^2}, \\
K_{34} = -\left[Bs_{11}\alpha_1^3 + (Bs_{12} + 2Bs_{66})\alpha_1\alpha_2^2 + \frac{As_{11}}{R}\alpha_1\right], K_{35} = -\left[Bs_{22}\alpha_2^3 + (Bs_{12} + 2Bs_{66})\alpha_1^2\alpha_2 + \frac{As_{12}}{R}\alpha_2\right], \\
K_{44} = Ass_{11}\alpha_1^2 + Ass_{66}\alpha_2^2 + Acc_{55}, \quad K_{45} = (Ass_{12} + Ass_{66})\alpha_1\alpha_2, \quad K_{55} = Ass_{22}\alpha_2^2 + Ass_{66}\alpha_1^2 + Acc_{55}
\end{gathered} \tag{25}$$

Elements of mass matrix [M]

$$\begin{aligned}
&M_{11} = \left(I_1 + 2\frac{I_2}{R} + \frac{I_3}{R^2}\right), \quad M_{12} = 0, M_{13} = -\left(I_2 + \frac{I_3}{R}\right)\alpha, \quad M_{14} = \left(I_4 + \frac{I_5}{R}\right), \quad M_{15} = 0, \\
&M_{22} = I_1, \quad M_{23} = -\left(I_2 + \frac{I_3}{R}\right)\beta, \quad M_{24} = 0, \quad M_{25} = I_4, \quad M_{33} = I_3\alpha^2 + I_3\beta^2 + I_1, \\
&M_{34} = -I_5\alpha, \quad M_{35} = -I_5\beta, \quad M_{44} = I_6, M_{45} = 0, M_{55} = I_6,
\end{aligned} \tag{26}$$

Elements of force vector {F}

$$\{F\} = \{0 \quad 0 \quad q_{mn} \quad 0 \quad 0\}^T \tag{27}$$

Elements of vector of unknowns { Δ }

$$\{\Delta\} = \{u_{mn} \quad v_{mn} \quad w_{mn} \quad \phi_{mn} \quad \psi_{mn}\}^T \tag{28}$$

Since the stiffness matrix [K] and mass matrix [M] are symmetric matrices, K_{ij}=K_{ji} and M_{ij}=M_{ji}

RESULTS AND DISCUSSION

In this paper, the present generalized displacement model is applied for the static and free vibration analysis of simply supported isotropic, laminated composite cylindrical shells. Numerical results for displacements, stresses and natural frequencies are obtained and compared with 3D elasticity solutions and other studies available in the literature. The following higher order and classical shell theories are recovered using the present generalized displacement model.

$$\text{PST [4]: } \zeta(z) = z\left[1-(4/3)(\bar{z})^2\right],\ \bar{z} = z/h$$

$$\text{TST [5]: } \zeta(z) = (h/\pi)\sin(\pi\bar{z})$$

$$\text{HST [6]: } \zeta(z) = z\cosh(1/2) - h\sinh(\bar{z})$$

$$\text{EST [7]: } \zeta(z) = z\, e^{-2(\bar{z})^2}$$

$$\text{FST [2]: } \zeta(z) = z$$

$$\text{CST [1]: } \zeta(z) = 0$$

Recovery of Transverse Shear Stresses

In multilayered shells, transverse shear stresses leads to the discontinuity at the layer interface when obtained using the constitutive relations. Therefore, in the present study, the transverse shear stresses are recovered by direct integration of stress equilibrium equations of theory of elasticity neglecting the body forces. The transverse shear stresses are recovered by layerwise integration of Eq. (29). The integration constants are determined by imposing the shear stress boundary conditions.

$$\tau_{xz}^{(k)} = -\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\left(\frac{\partial\sigma_x^{(k)}}{\partial x} + \frac{\partial\tau_{xy}^{(k)}}{\partial y}\right)dz + C_1 \text{ and } \tau_{yz}^{(k)} = -\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\left(\frac{\partial\sigma_y^{(k)}}{\partial y} + \frac{\partial\tau_{xy}^{(k)}}{\partial x}\right)dz + C_2 \tag{29}$$

Numerical Results

The non-dimensional numerical results are presented in Tables 3 through 8 and graphically in Figures 2 through 4. The material properties and non-dimensional parameters considered in the numerical examples are presented in Table 2.

Table 2. Material properties considered in numerical examples from Bhimaraddi and Chandrashekhara [37]

Examples	Material properties	Non-dimensional parameters
1	$E = 210\ GPa,\ \mu = 0.3$	$\bar{w} = \frac{wE}{q_0 a},\ \bar{\sigma}_x = \frac{\sigma_x}{q_0},\ \bar{\tau}_{zx} = \frac{\tau_{zx}}{q_0}$
2, 3	$E_1 = 36.0885,\ E_2 = 26.2818,\ G_{12} = 4.9033,$ $G_{31} = 4.4130,\ G_{23} = 4.0208,\ \mu_{12} = \mu_{13} = \mu_{23} = 0.105$	$\bar{w} = \frac{wE_1}{q_0 a},\ \bar{\sigma}_x = \frac{\sigma_x}{q_0},\ \bar{\tau}_{zx} = \frac{\tau_{zx}}{q_0}$
4	$E_1/E_2 = 25,\ E_2 = 26.2818,\ G_{23}/E_2 = 0.2,$ $G_{12}/E_2 = G_{31}/E_2 = 0.5,\ \mu_{12} = \mu_{13} = \mu_{23} = 0.25$	$\bar{\omega} = \omega\left(a^2/h\right)\sqrt{\rho/E_2}$

Table 3. Displacements and stresses in isotropic cylindrical shells with different *R/a* values (*h/a*=0.1, *a/b*=1)

R/a	Model	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
5	Present	PST	28.756	20.808	2.3132
	Present	TST	28.753	20.820	2.3125
	Present	HST	28.756	20.807	2.3133
	Present	EST	28.760	20.860	2.3162
	Present	FST	28.508	20.620	2.3193
	Bhimaraddi and Chandrashekhara [37]	3D Elasticity	29.003	21.138	2.3455

Table 3. Displacements and stresses in isotropic cylindrical shells with different *R/a* values (*h/a*=0.1, *a/b*=1) (cont.)

R/a	Model	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
10	Present	PST	29.390	20.532	2.3642
	Present	TST	29.386	20.544	2.3635
	Present	HST	29.390	20.532	2.3642
	Present	EST	29.394	20.585	2.3672
	Present	FST	29.130	20.342	2.3699
	Bhimaraddi and Chandrashekhara [37]	3D Elasticity	29.379	20.719	2.3787
20	Present	PST	29.552	20.277	2.3773
	Present	TST	29.549	20.288	2.3766
	Present	HST	29.552	20.277	2.3773
	Present	EST	29.557	20.330	2.3803
	Present	FST	29.290	20.088	2.3830
	Bhimaraddi and Chandrashekhara [37]	3D Elasticity	29.445	20.413	2.3847

Example 1: Static Analysis of Homogenous Isotropic Cylindrical Shells

Table 3 shows comparison of transverse displacements and stresses of homogenous isotropic cylindrical shell subjected to sinusoidal transverse load. The non-dimensional results are presented for various *R/a* (=5, 10, 20) values. Table 2 shows material properties and non-dimensional parameters for this example. The numerical results are presented by using higher order shell theories recovered from the present generalized displacement model and compared with three-dimensional (3D) elasticity solutions presented by Bhimaraddi and Chandrashekhara [37]. From Table 3, it is observed that the trigonometric shell theory (TST) shows transverse displacements in close agreement with 3D elasticity solution for *R/a*=10 and 20, whereas, exponential shell theory (EST) shows transverse displacements slightly on higher side. The parabolic shell theory (PST) and hyperbolic shell theory (HST) show identical results of displacements and stresses. The first order shell theory underestimates the transverse displacements and stresses for all *R/a* values.

Table 4. Displacements and stresses in orthotropic cylindrical shells with different *R/a* values (*h/a*=0.1, *a/b*=1)

R/a	Model	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
5	Present	PST	54.434	27.914	2.5736
	Present	TST	54.425	27.944	2.5721
	Present	HST	54.480	27.831	2.5656
	Present	EST	54.446	28.052	2.5798
	Present	FST	53.609	27.449	2.5876
	Present	CST	49.442	27.515	---
	Bhimaraddi and Chandrashekhara [37]	3D Elasticity	53.298	28.466	2.6283
10	Present	PST	55.384	27.820	2.6185
	Present	TST	55.374	27.850	2.6170
	Present	HST	55.432	27.735	2.6104
	Present	EST	55.396	27.960	2.6249
	Present	FST	54.530	27.348	2.6320
	Present	CST	50.224	27.423	---
	Bhimaraddi and Chandrashekhara [37]	3D Elasticity	54.958	28.154	2.5473
20	Present	PST	55.627	27.649	2.6300
	Present	TST	55.617	27.680	2.6285
	Present	HST	55.675	27.564	2.6218
	Present	EST	55.639	27.790	2.6364
	Present	FST	54.765	27.179	2.6434
	Present	CST	50.424	27.267	---
	Bhimaraddi and Chandrashekhara [37]	3D Elasticity	55.023	27.908	2.6554

Example 2: Static Analysis of Homogenous Orthotropic Cylindrical Shells

Non-dimensional displacements and stresses of homogenous orthotropic cylindrical shells subjected to sinusoidal load are presented in Table 4 and compared with 3D elasticity solution of Bhimaraddi and Chandrashekhara [37]. The numerical results are presented for different *R/a* values. Material properties and non-dimensional parameter used to obtain numerical results for this example are presented in Table 2. Examination of Table 4 reveals that the transverse displacements predicted by FST are in good agreement with those presented by Bhimaraddi and Chandrashekhara [37] in 3D elasticity solution whereas EST predicts stresses in excellent agreement with those presented in 3D elasticity solution. Non-dimensional value of transverse displacements increases with respect to increase in *R/a* values whereas stresses are decreases with increase in *R/a* values.

Table 5. Displacements and stresses in laminated cylindrical shells (*R/a* =1, *h/a*=0.1, *a/b*=1)

Model	Theory	$0^0/90^0$			$0^0/90^0/0^0$		
		$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
Present	PST	35.358	17.198	1.4140	35.097	21.246	1.6189
Present	TST	35.362	17.213	1.4134	35.094	21.264	1.6179
Present	HST	35.343	17.761	1.4139	35.079	21.300	1.6190
Present	EST	35.223	17.229	1.4128	35.103	21.333	1.6176
Present	FST	35.020	17.030	1.4284	34.743	21.031	1.6382
Present	CST	33.247	17.030	---	32.938	21.031	---
Bhimaraddi and Chandrashekhara [37]	3D Elasticity	41.340	23.700	1.7763	40.823	23.987	1.8290

Example 3: Static Analysis of Laminated Cylindrical Shells

In this example, all the higher order shell theories recovered from the present generalized displacement model are applied for the static analysis of laminated composite cylindrical shells on rectangular planform. Material properties and non-dimensional parameters are presented in Table 2. In case of laminated shells, it is assumed that all layers are of equal thickness i.e. for two-ply ($0^0/90^0$) laminated shells thickness of each layer is *h*/2 whereas for three-ply ($0^0/90^0/0^0$) laminated shells thickness of each layer is *h*/3. The present results are compared with 3D elasticity solution presented by Bhimaraddi and Chandrashekhara [37]. The numerical results are presented in Table 5 for two-ply ($0^0/90^0$) and three-ply ($0^0/90^0/0^0$) laminated shells for *R/a* =1, *h/a*=0.1 and *a/b*=1. For both the lamination schemes, TST predicts transverse displacement in close agreement with 3D elasticity solution. The FST and CST underpredict the displacements and stresses for both the types of laminated shells. Through-the-thickness distributions of in-plane normal and transverse shear stresses are plotted in Figures 2 and 3 for two-ply ($0^0/90^0$) and three-ply ($0^0/90^0/0^0$) laminated shells respectively. Both the stresses are observed to be maximum in 0 degree layers and minimum in 90 degree layer. The maximum in-plane normal stresses are observed at the bottom of shell i.e. (z=h/2) whereas maximum transverse shear stresses are observed at the middle surface of the shell.

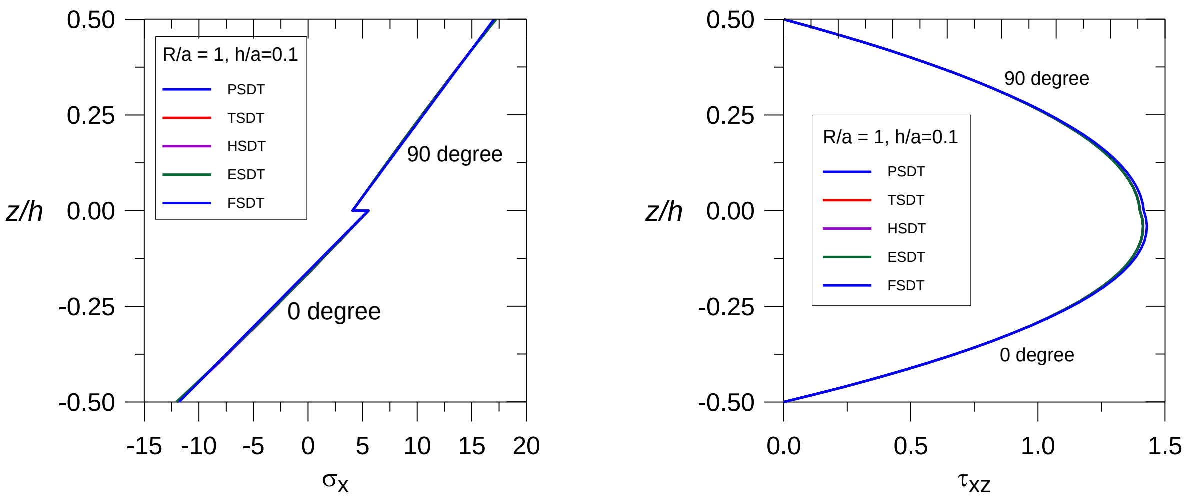

Figure 2. Through-the-thickness profiles of stresses for two-ply ($0^0/90^0$) laminated shells

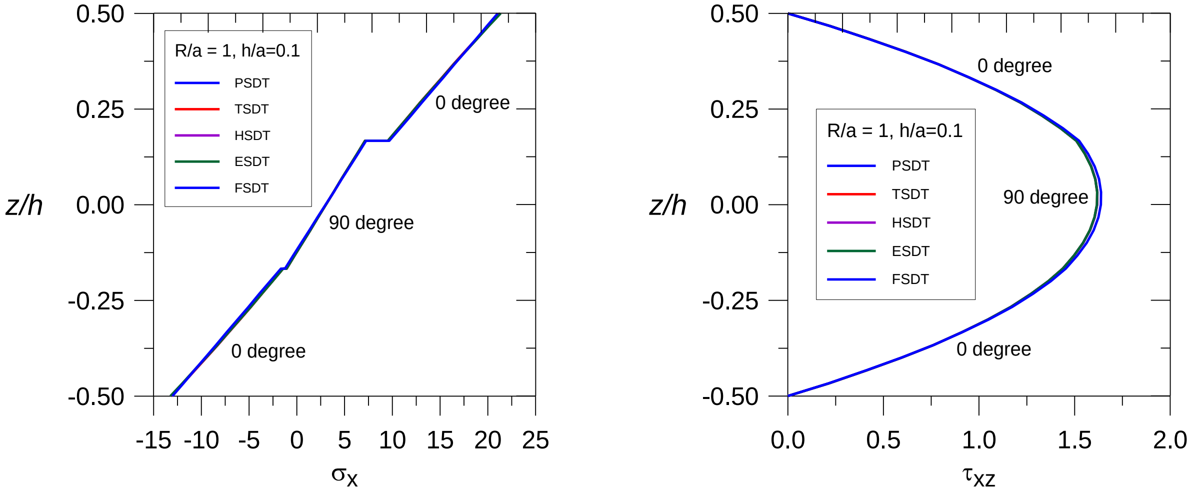

Figure 3. Through-the-thickness profiles of stresses for three-ply ($0^0/90^0/0^0$) laminated shells

Table 6. Natural frequencies in cylindrical shells with different *R/a* values (*h/a*=0.1, *a/b*=1)

R/a	Model	Theory	Isotropic	Orthotropic	$0^0/90^0$	$0^0/90^0/0^0$
0.5	Present	PST	10.0200	14.0875	14.2221	15.2534
	Present	TST	10.0199	14.0847	14.2240	15.2282
	Present	HST	10.0205	14.0975	14.2275	15.2557
	Present	EST	10.0211	14.1178	14.2400	15.2417
	Present	FST	10.0302	14.2902	14.2538	15.6721
	Asadi et al. [24]	FSDTQ	9.59610	--	13.7710	15.2500
	Asadi et al. [24]	3D-FEM	7.55300	--	13.7720	14.8400
1	Present	PST	7.34706	13.0226	10.9307	12.9914
	Present	TST	7.34689	13.0188	10.9345	12.9541
	Present	HST	7.34782	13.0368	10.9390	12.9948
	Present	EST	7.34898	13.0655	10.9585	12.9742
	Present	FST	7.36499	13.3080	10.9594	13.6064
	Asadi et al. [24]	FSDTQ	7.09000	--	10.6660	13.1870
	Asadi et al. [24]	3D-FEM	7.08100	--	10.6860	12.5900
2	Present	PST	6.22746	12.6228	9.57321	12.1277
	Present	TST	6.22725	12.6185	9.57804	12.0849
	Present	HST	6.22841	12.6386	9.58313	12.1316
	Present	EST	6.22990	12.6707	9.60642	12.1080
	Present	FST	6.25008	12.9410	9.60041	12.8294
	Asadi et al. [24]	FSDTQ	6.09130	--	9.45770	12.4430
	Asadi et al. [24]	3D-FEM	6.09210	--	9.48550	11.7690

Table 7. First five natural frequencies in two-ply ($0^0/90^0$) laminated cylindrical shells with different *R/a* values (*h/a*=0.1, *a/b*=1)

R/a	Model	Theory	ω_1	ω_2	ω_3	ω_4	ω_5
0.5	Present	PST	14.2221	22.6425	29.8696	32.5766	40.3561
	Present	TST	14.2240	22.6697	29.8837	32.6103	40.4125
	Present	HST	14.2275	22.7138	29.9156	32.6829	40.6236
	Present	EST	14.2400	22.9038	30.0395	32.9716	41.4293
	Present	FST	14.2538	22.7374	29.9785	32.7823	40.4045
	Asadi et al. [24]	FSDTQ	13.7710	21.0370	29.5710	31.2000	38.0730
	Asadi et al. [24]	3D-FEM	13.7720	21.0400	29.6390	31.4110	38.2660

Table 7. First five natural frequencies in two-ply ($0^0/90^0$) laminated cylindrical shells with different *R/a* values (*h/a*=0.1, *a/b*=1) (cont.)

R/a	Model	Theory	ω_1	ω_2	ω_3	ω_4	ω_5
1	Present	PST	10.9307	22.3397	24.5762	31.1716	40.3531
	Present	TST	10.9345	22.3691	24.5999	31.2102	40.4138
	Present	HST	10.9390	22.4141	24.6417	31.2855	40.6210
	Present	EST	10.9585	22.6127	24.8177	31.5948	41.2947
	Present	FST	10.9594	22.4171	24.6705	31.3472	40.3429
	Asadi et al. [24]	FSDTQ	10.6660	21.7050	24.0900	30.3680	38.7220
	Asadi et al. [24]	3D-FEM	10.6860	21.7670	24.1910	30.6140	38.8960
2	Present	PST	9.57321	22.1518	22.6620	30.6846	40.2263
	Present	TST	9.57804	22.1818	22.6901	30.7248	40.2888
	Present	HST	9.58313	22.2267	22.7350	30.8001	40.4933
	Present	EST	9.60642	22.4266	22.9309	31.1130	41.2947
	Present	FST	9.60041	22.2207	22.7412	30.8446	40.1893
	Asadi et al. [24]	FSDTQ	9.45770	21.6760	22.1500	29.9590	38.6080
	Asadi et al. [24]	3D-FEM	9.48550	21.7430	22.2460	30.1930	38.7450

Example 4: Free Vibration Analysis of Homogenous and Laminated Cylindrical Shells

In this example, the present theories are applied for the free vibration analysis of homogenous and laminated cylindrical shells. Natural frequencies are presented in Tables 6 through 8. Material properties and non-dimensional form are presented in Table 2. Isotropic material properties are similar to example 1. Table 6 shows first natural frequencies in isotropic, orthotropic and laminated composite cylindrical shells whereas Tables 7 and 8 represents first five natural frequencies of two-ply ($0^0/90^0$) and three-ply ($0^0/90^0/0^0$) laminated cylindrical shells. Since 3D elasticity solution for frequency analysis is not available in the literature, the present results are compared with those presented by Asadi et al. [24]. Tables 6 through 8 show that the natural frequencies predicted by the present models are in good agreement with those presented by Asadi et al. [24] for all cases. Examination of these Tables reveals that the natural frequencies are decreases with increase in *R/a* values.

Table 8. First five natural frequencies in three-ply ($0^0/90^0/0^0$) laminated cylindrical shells with different *R/a* values (*h/a*=0.1, *a/b*=1)

R/a	Model	Theory	ω_1	ω_2	ω_3	ω_4	ω_5
0.5	Present	PST	15.2534	27.8837	28.6315	34.0363	42.8207
	Present	TST	15.2282	27.8650	28.4831	33.9086	42.7973
	Present	HST	15.2557	27.8850	28.6434	34.0458	42.8216
	Present	EST	15.2417	27.8754	28.7184	34.1171	42.8107
	Present	FST	15.6721	28.2423	31.9990	37.0829	43.4225
	Asadi et al. [24]	FSDTQ	15.2500	17.7890	29.4910	34.7950	34.9130
	Asadi et al. [24]	3D-FEM	14.8400	17.4680	29.0940	32.4640	33.0460
1	Present	PST	12.9914	21.8328	29.1613	33.2639	34.8957
	Present	TST	12.9541	21.8041	29.0050	33.1243	34.8644
	Present	HST	12.9948	21.8352	29.1734	33.2742	34.8980
	Present	EST	12.9742	21.8213	29.2523	33.3525	34.8878
	Present	FST	13.6064	22.3960	32.7008	35.7787	36.5792
	Asadi et al. [24]	FSDTQ	13.1870	18.5240	30.5640	32.2320	34.5230
	Asadi et al. [24]	3D-FEM	12.5900	18.0050	29.7320	30.1890	32.0370
2	Present	PST	12.1277	19.4390	29.3036	29.3036	33.0494
	Present	TST	12.0849	19.4049	29.1453	29.1453	32.9065
	Present	HST	12.1316	19.4419	29.3160	29.3160	33.0598
	Present	EST	12.1080	19.4260	29.3960	29.3960	33.1402
	Present	FST	12.8294	20.1085	32.8892	32.8892	36.4397
	Asadi et al. [24]	FSDTQ	12.4430	18.6770	30.8390	31.3230	34.4560
	Asadi et al. [24]	3D-FEM	11.7690	18.1590	28.6000	30.4710	31.9280

CONCLUSION

In this study, several higher order and classical shell theories are recovered using a generalized displacement model for the static and free vibration analysis of laminated composite cylindrical shells on rectangular planform. The differential equations of the present generalized displacement model are derived by using the principle of virtual work and further solved analytically using the Navier's solution technique. Transverse shear stresses are recovered from 3D equilibrium equations of theory of elasticity. Numerical results obtained using all the present models are compared with 3D elasticity solution and available literature and are found in good agreement. The numerical results and their comparison proved the validity and accuracy of the present generalized displacement model.

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