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ORIGINAL ARTICLE

Quintic B-spline collocation method for numerical solution of free vibration of tapered Euler-Bernoulli beam on variable Winkler foundation

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ABSTRACT – The collocation method is the method for the numerical solution of integral equations and partial and ordinary differential equations. The main idea of this method is to choose a number of points in the domain and a finite-dimensional space of candidate solutions. So, that solution satisfies the governing equation at the collocation points. The current paper involves developing, and a comprehensive, step-by step procedure for applying the collocation method to the numerical solution of free vibration of tapered Euler-Bernoulli beam. In this stusy, it is assumed the beam rested on variable Winkler foundation. The simplicity of this approximation method makes it an ideal candidate for computer implementation. Finally, the numerical examples are introduced to show efficiency and applicability of quintic B-spline collocation method. Numerical results are demonstrated that quintic B-spline collocation method is very competitive for numerical solution of frequency analysis of tapered beam on variable elastic foundation.

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INTRODUCTION

Many engineering problems can be idealized as a beam on foundation. Winkler, Kerr, Pasternak, Viscoelastic, Vlasov and Hetenyi models are different types of foundation that can be used in modelling foundation in these problems [1]. The most common model is Winkler foundation. However, the Winkler model is inadequate for modelling of soil in various problems [2]. The quintic B-spline collocation method is used to numerical solution of free vibration of tapered damped Euler-Bernoulli (EB) beam on foundation in this paper. A spline function is a piecewise polynomial function in a variable x. The spline function is the composite of several internal point that number points must greater than or equal to $(k+1)$ degree. The B-spline functions of $(k+1)$ degree is used to solve the differential equation with (k) degree [3]. Also, highest grade of B-spline function recursively comes of B-spline functions with lower grade.

Over the years, collocation method is applied to solve differential equations with different boundary conditions. An overview of the formulation, analysis and implementation of orthogonal spline collocation is provided for numerical solution of differential equations in two space variables by Bialecki and Fairweather [4]. The sextic B-spline function for numerical solution of a system of second-order boundary value problems is presented by Rashidinia et al. [5]. In this paper, the results are compared with the finite difference method (FDM) and it is demonstrated the results using the sextic B-spline collocation method are better than the FDM. The quartic B-spline collocation method is applied for numerical solution of Burgers' equation by Saka and Dağ [6]. Also, two-parameter singularly perturbed boundary value problem is solved using the B-spline method by Kadalbajoo and Yadaw [7]. In this paper, it is shown that the convergence analysis is a uniform convergence of second order. Quintic nonpolynomial B-spline collocation for a fourth-order boundary value problem is investigated by Ramadan et al. [8]. The results are shown that the quintic nonpolynomial B-spline collocation method presents better approximations. On the other hand, the presented method generalized all existing polynomial B-spline methods up to fourth-order.

Hsu applied B-spline collocation method for estimated the free vibration of non-uniform EB with typical boundary conditions on a uniform foundation [9]. The boundary conditions that accompanied the spline collocation method are used to convert the partial differential equations of non-uniform EB vibration problem into a discrete eigenvalue problem. Aziz and Šarler proposed the uniform Haar wavelets for the second-order boundary-value problems [10]. The isogeometric collocation method is presented for analysis of Timoshenko beam by Da Veiga et al. [11]. Zarebnia and Parvaz solved the Kuramoto-Sivashinsky equation using septic B-spline collocation method [12]. A linear combination of these functions is used to approximate solution. In this paper, using the Von-Neumann stability analysis technique, it is shown that the septic B-spline collocation method is unconditionally stable. Also, cubic B-spline collocation method is used to find the solution of the problem arising from chemical reactor theory [13]. The sextic B-spline collocation method is applied to find the numerical solution of the problem with the partial differential equation by Mohammadi [14]. The convergence analysis for present approximation is explored in details for EB with cantilever and fixed boundary conditions. The isogeometric collocation method is applied to solution of thin structural problems that describe using the EB and Kirchhoff plate models by Reali and Gomez [15]. Also, the isogeometric method is used for numerical solution

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of plate problems that describe using Reissner–Mindlin plate model by Kiendl et al. [16]. On the other hand, Akram applied the sextic B-spline collocation method for solving of boundary value problems with fifth-order [17]. The numerical results are shown that the presented method developed is better than quartic spline method. The isogeometric collocation method is used to analysis of spatial rods by Auricchio et al. [18]. The convergence and stability analysis are shown using the theoretical analysis in this paper. Liu and Li applied the energetic boundary functions collocation method for composite beams [19]. Also, the approximate solution based on collocation method is presented for the boundary value problem [20]. Chebyshev wavelet collocation method is used by Çelik for the non-uniform Euler–Bernoulli beam [21]. In this study, the beam is assumed with various supporting conditions. Kiendl et al. develop the isogeometric collocation method for the Timoshenko beam [22]. Also, frequancy analysis of graded tapered beam is presented using chebyshev collocation method by Chen [23].

In this study, analysis of elastically restrained tapered EB on variable Winkler foundation is presented using quintic B-spline collocation method. In the other hand, a damped EB on variable Winkler foundation is presented in a general form. In this paper, the main objective is to introduce a practical numerical solution based on quintic B-spline collocation method for elastically restrained tapered damped EB rested on variable Winkler foundation. For this propose, in section 2, the quintic B-spline collocation method outlines. Then, the presented method is applied to the frequancy analysis of tapered damped EB rested on the variable Winkler foundation in section 3. So, section 4 explains various numerical examples to display applicability and efficiency of presented method. Finally, conclusions are introduced in section 5, briefly.

QUINTIC B-SPLINE COLLOCATION METHOD

Let $x = (x_0, x_1, \dots, x_N)$ be knot vector. The k degree B-spline function can be given as follows [24]:

$$B_{k,i}(x) = \frac{x - x_i}{x_{(i+k)} - x_i} B_{(k-1),i}(x) + \frac{x_{(i+k+1)} - x}{x_{(i+k+1)} - x_{(i+1)}} B_{(k-1),(i+1)}(x) \tag{1}$$

and for $k = 0$, the B-spline function is determined as follow:

$$B_{0,i} = \begin{cases} 1 & for \quad x \in [x_i, x_{(i+1)}) \\ 0 & otherwise \end{cases} \tag{2}$$

where $0 \le i \le N - k - 1$ and $1 \le k \le N - 1$. Therefore, the quintic B-spline function can be given using Eq. (1). The quintic B-spline function, $B_{5,i}(x)$ is as follows:

$$B_i(x) = \frac{1}{120h^5} \begin{cases} (x - x_i + 3h)^5 & x \in [x_{(i-3)}, x_{(i-2)}) \\ (x - x_i + 3h)^5 - 6(x - x_i + 2h)^5 & x \in [x_{(i-2)}, x_{(i-1)}) \\ (x - x_i + 3h)^5 - 6(x - x_i + 2h)^5 + 15(x - x_i + h)^5 & x \in [x_{(i-1)}, x_i) \\ (-x + x_i + 3h)^5 - 6(-x + x_i + 2h)^5 + 15(-x + x_i + h)^5 & x \in [x_i, x_{(i+1)}) \\ (-x + x_i + 3h)^5 - 6(-x + x_i + 2h)^5 & x \in [x_{(i+1)}, x_{(i+2)}) \\ (-x + x_i + 3h)^5 & x \in [x_{(i+2)}, x_{(i+3)}) \\ 0 & otherwise \end{cases} \tag{3}$$

The solution domain is $0 \le x \le L$ in this paper. This domain is divided into N segments with length $h = L/N$, by the knots x_i where $i = 0, 1, \dots, N$ and $0 = x_0 < x_1 < \dots < x_N = L$. In quintic B-spline collocation method, basic function is defined as follows:

$$y(x) = \sum_{i=-2}^{N+2} c_i B_i(x) \tag{4}$$

where $B_{-2}(x), \dots, B_{N+2}(x)$ are the quintic B-spline functions at knots. Also, $c_{-2}, \dots, c_{N+2}$ are unknown coefficients that can be determined using the collocation form of the governing differential equation of the tapered damped EB rested on variable Winkler foundation and boundary conditions at each end of the beam. On the other hand, 1st, 2nd, 3rd and 4td derivatives of $B_i(x)$ respect to variable x may be used in the governing differential equation of the EB. Table 1 is presented the values of $B_i(x)$ and its derivatives at the nodal points.

MODELLING OF TAPERED DAMPED EB ON FOUNDATION

A tapered damped EB on variable Winkler foundation is considered in this paper. This beam is restrained against rotation and translation at its ends, as shown in Figure 1. K_{RL} and K_{RR} are rotational coefficients at left and right edges, respectively. Also, K_{TL} and K_{TR} are transverse coefficients at left and right edges, respectively. For the free vibration of the tapered damped EB on variable Winkler foundation, the governing differential equation can be given by:

$$\left(EIW_{,xxxx}+2EI_{,x}W_{,xxx}+EI_{,xx}W_{,xx}\right)+\left(r_iW_{,xxxxt}+2r_{i,x}W_{,xxxt}+r_{i,xx}W_{,xxt}\right)+\rho AW_{,tt}+r_eW_{,t}+K_fW=0 \quad (5)$$

where W is transverse deflection of EB, $A(x)$ and $I(x)$ denote the cross section function and moment of inertia function (at x position), respectively. Also, E, r_i, r_e and ρ present the Young's modulus, internal damping coefficient of damped EB, it is generally very small [25], the external damping coefficient of damped EB, and material density, respectively. In this paper, it is assumed that the Winkler foundation modulus ($K_f(x)$) through the EB length can vary constantly or linearly. Therefore, the $K_f(x)$ is given below:

$$K_f(x)=K_{f0}(1-\beta x) \quad (6)$$

where K_{f0} is the foundation modulus at $x=0$ and β is the variation parameters. The damping effects are assumed to be proportional to the stiffness properties of beam for internal damping and mass of beam for external damping in this study, respectively. Therefore, these damping can be considered as [26]:

$$r_i(x)=\alpha_i\, E\, I(x) \quad (7)$$

$$r_e(x)=\alpha_e\, \rho A(x) \quad (8)$$

where in these equation, α_i and α_e are proportionality constants. On the other hand, the dimensionless damping ratio of damped EB with internal and external damping can be represented as (for n^{th} mode):

Table 1. $B_i(x)$ and its derivatives at nodal points

	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}
B_i	0	$\frac{1}{120}$	$\frac{26}{120}$	$\frac{66}{120}$	$\frac{26}{120}$	$\frac{1}{120}$	0
hB_i'	0	$\frac{5}{120}$	$\frac{50}{120}$	0	$-\frac{50}{120}$	$-\frac{5}{120}$	0
h^2B_i''	0	$\frac{20}{120}$	$\frac{40}{120}$	-1	$\frac{40}{120}$	$\frac{20}{120}$	0
h^3B_i'''	0	$\frac{60}{120}$	-1	0	1	$-\frac{60}{120}$	0
$h^5B_i^{(4)}$	0	1	-4	6	-4	1	0

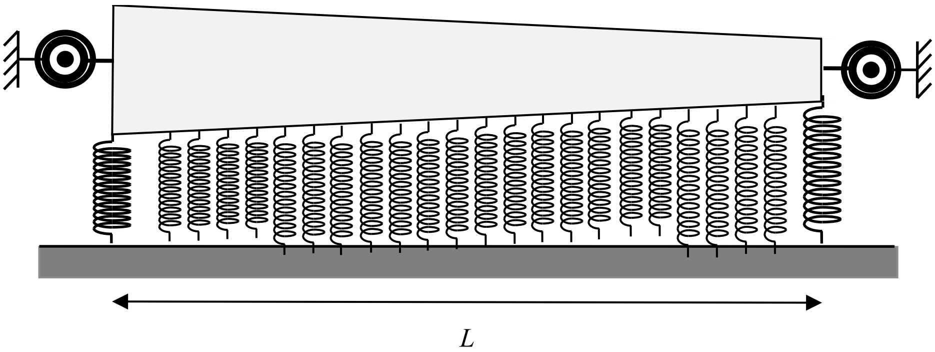


Figure 1. Tapered damped EB with general boundary conditions resting on Winkler foundation

$$\xi_i = \frac{1}{2}\alpha_i \, \omega_n \tag{9}$$

$$\xi_e = \frac{1}{2}\frac{\alpha_e}{\omega_n} \tag{10}$$

The transverse deflection of EB can be considered as:

$$W(x,t) = w(x) Exp(i\,\omega t) \tag{11}$$

where ω is circular frequency of tapered EB. In addition, $w(x)$ is deflection amplitude of the beam. So, by substituting Eq. (11) into Eq. (5), the governing differential equation result as:

$$(1+\mathrm{i}\alpha_i\omega)\left(EIW_{,xxxx} + 2EI_{,x}W_{,xxx} + EI_{,xx}W_{,xx}\right) + \left(K_f - \rho\omega(\omega - \mathrm{i}\alpha_e)A\right)W = 0 \tag{12}$$

Based on EB theory, general boundary conditions are given below [22]:

$$\begin{aligned} M(0,t) &= K_{RL}\,\theta(0,t) & V(0,t) &= -K_{TL}W(0,t) \\ M(L,t) &= -K_{RR}\,\theta(L,t) & V(L,t) &= K_{TR}W(L,t) \end{aligned} \tag{13}$$

where M, V and θ are the bending moment ($M = EI\dfrac{\partial^2 W}{\partial x^2}$), the shear force ($\theta = \dfrac{\partial}{\partial x}\left(EI\dfrac{\partial^2 W}{\partial x^2}\right)$), and the slope ($\theta = \dfrac{\partial W}{\partial x}$), respectively. Standard boundary conditions for EB theory are presented in Table 2. For this study, quintic B-spline collocation method is used to analyze tarped damped EB. By applying quintic B-spline function, Eq. (12) can be written as the following for $j = 0,1,\ldots,N$:

$$\begin{aligned} &(1+\mathrm{i}\alpha_i\omega)\left(EI(x_j)\sum_{i=-2}^{N+2} c_i B_i^{(4)}(x_j) + 2EI'(x_j)\sum_{i=-2}^{N+2} c_i B_i^{(3)}(x) + EI''(x_j)\sum_{i=-2}^{N+2} c_i B_i^{(2)}(x_j)\right) + \\ &\qquad \left(K_f(x_j) - \rho\omega(\omega - \mathrm{i}\alpha_e)A(x_j)\right)\sum_{i=-2}^{N+2} c_i B_i(x_j) = 0 \end{aligned} \tag{14}$$

From Table 1 and Eq. (4), y_i, y_i', y_i'', y_i''', and $y_i^{(4)}$ at node points are obtained as follows:

Table 2. Standard boundary conditions for EB theory

Support	Boundary conditions
Free	$\frac{\partial^2 W}{\partial x^2} = \frac{\partial}{\partial x}\left(EI\,\frac{\partial^2 W}{\partial x^2}\right) = 0$
Simply supported	$W = \frac{\partial^2 W}{\partial x^2} = 0$
Sliding	$\frac{\partial W}{\partial x} = \frac{\partial}{\partial x}\left(EI\,\frac{\partial^2 W}{\partial x^2}\right) = 0$
Clamped	$W = \frac{\partial W}{\partial x} = 0$

$$
\begin{aligned}
y_i &= \frac{1}{120}(c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}) \\
y_i' &= \frac{1}{120h}(5c_{i-2} + 50c_{i-1} - 50c_{i+1} - 5c_{i+2}) \\
y_i'' &= \frac{1}{120h^2}(20c_{i-2} + 40c_{i-1} - 120c_i + 40c_{i+1} + 20c_{i+2}) \\
y_i''' &= \frac{1}{120h^3}(60c_{i-2} - 120c_{i-1} + 120c_{i+1} - 60c_{i+2}) \\
y_i^{(4)} &= \frac{1}{120h^4}(120c_{i-2} - 480c_{i-1} + 720c_i - 480c_{i+1} + 120c_{i+2})
\end{aligned}
\tag{15}
$$

Therefore, Eq. (14) can be stated as for $j = 0,1,\ldots,N$:

$$
\begin{aligned}
&\left(20\lambda_1\left(6I(x_j) + 6h\,I'(x_j) + h^2 I''(x_j)\right) + \left(K_f(x_j) - \lambda_2 A(x_j)\right)\right) c_{i-2} + \\
&\left(40\lambda_1\left(-12I(x_j) - 6h\,I'(x_j) + h^2 I''(x_j)\right) + 26\left(K_f(x_j) - \lambda_2 A(x_j)\right)\lambda_2\right) c_{i-1} + \\
&\left(120\lambda_1\left(6I(x_j) - h^2 I''(x_j)\right) + 66\left(K_f(x_j) - \lambda_2 A(x_j)\right)\lambda_2\right) c_i + \\
&\left(40\lambda_1\left(-12I(x_j) + 6h\,I'(x_j) + h^2 I''(x_j)\right) + 26\left(K_f(x_j) - \lambda_2 A(x_j)\right)\lambda_2\right) c_{i+1} + \\
&\left(20\lambda_1\left(6I(x_j) - 6h\,I'(x_j) + h^2 I''(x_j)\right) + \left(K_f(x_j) - \lambda_2 A(x_j)\right)\lambda_2\right) c_{i+2} = 0
\end{aligned}
\tag{16}
$$

where:

$$
\lambda_1 = \frac{E}{h^4}\left(1 + \mathrm{i}\alpha_i\omega\right)
$$

$$
\lambda_2 = \rho\,\omega\left(\omega - \mathrm{i}\alpha_e\right)
$$

The system in Eq. (16) is the set of $N+1$ equations with $N+5$ unknowns. The boundary conditions at each end present four extra equations. Thus, these equations depend on end support. For example, these equations for simply supported condition can be given as:

$$c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2 = 0 \tag{17a}$$

$$c_{-2} + 2c_{-1} - 6c_0 + 2c_1 + c_2 = 0 \tag{17b}$$

$$c_{N-2} + 2c_{N-1} - 6c_N + 2c_{N+1} + c_{N+2} = 0 \tag{17c}$$

$$c_{N-2} + 26c_{N-1} + 66c_N + 26c_{N+1} + c_{N+2} = 0 \tag{17d}$$

Finally, N+5 algebraic equations are obtained. $c_{-2}, c_{-1}, \ldots, c_{N+1}, c_{N+2}$ and ω are the unknown variables in these algebraic equations. Therefore, the matrix equation is given as:

$$\begin{bmatrix} A_{-2,-2} & A_{-2,-1} & A_{-2,0} & . & . & A_{-2,N+2} \\ A_{-1,-2} & A_{-1,-1} & A_{-1,0} & . & . & A_{-1,N+2} \\ A_{0,-2} & A_{0,-1} & A_{0,0} & . & . & A_{0,N+2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{N+2,-2} & A_{N+2,-1} & A_{N+2,0} & . & . & A_{N+2,N+2} \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ . \\ . \\ c_{N+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} \tag{18}$$

In matrix Eq. (18), the nontrivial solution, when determinant of coefficients is equal to zero, is acquired. The coefficients matrix determinant is the associated frequency equation. In this paper, the numerical solution for frequency analysis of tapered damped EB that is generated using proposed method has a general form.

NUMERICAL RESULTS

To evaluate efficiency of proposed method, it is applied for solving the some examples. Also, the numerical computations are performed by the WOLFRAM MATHEMATICA software.

Uniform EB with Standard Boundary Conditions

To demonstrate accuracy of presented solution, an uniform EB is assumed with standard boundary conditions. The frequency parameters ($\varphi = \omega\sqrt{\rho A L^4 / EI}$) of the uniform EBs using the presented method along with the power series method [27] and the exact solution [28] and are compared in Table 3. Results are presented that the maximum difference is 1.74% for N=25 and 0.44% for N=50, hence, they are fairly close. Convergence of first five frequencies are demonstrated in Figure 2. The first natural frequancy of the uniform EB is less sensitive to number of terms. In addition, the maximum difference of the frequency parameter for clamped-free EB is approximately 12.41% for N=5 and 0.19% for N=60.

Cantilever Tapered EB on Uniform Winkler Foundation

In this example, effect of elastic Winkler foundation on frequency parameters of the cantilever tapered EB is presented. The tapered beam characteristics is considered as:

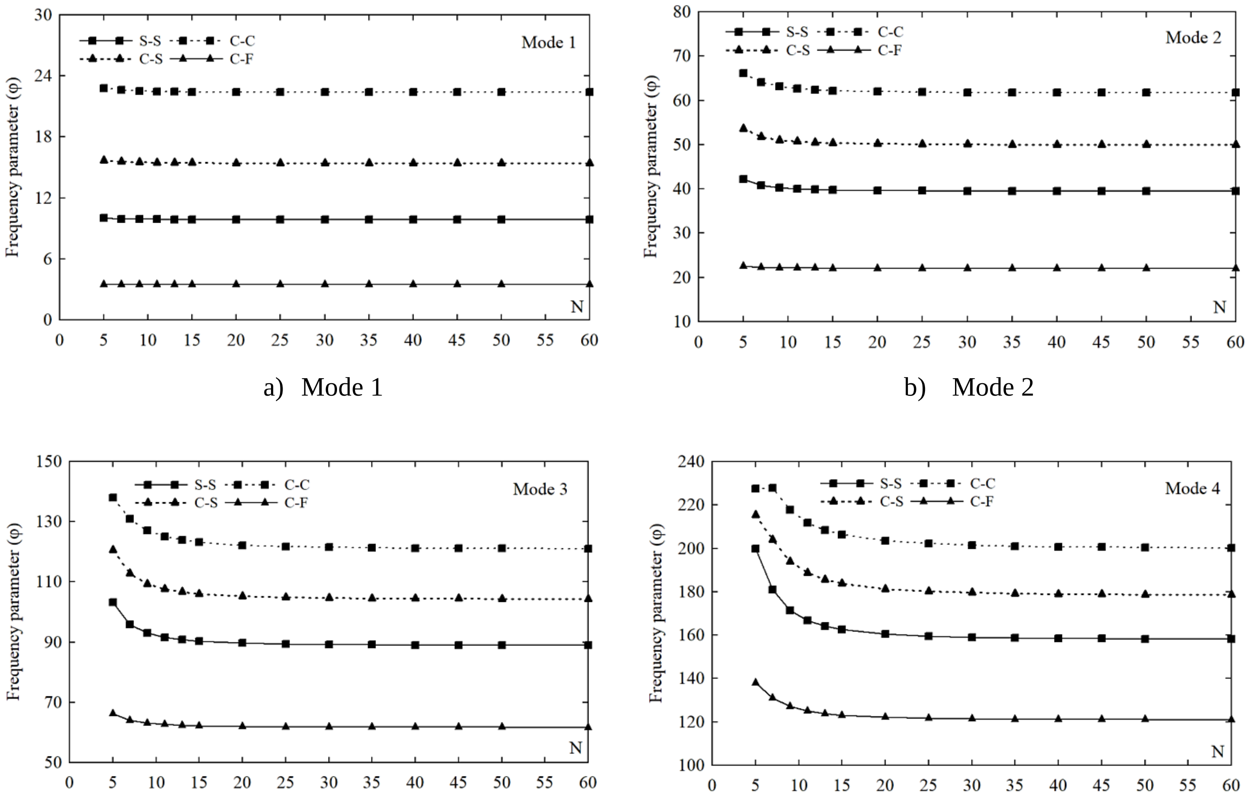
$$A(x) = A_0\left(1 - 0.5\frac{x}{L}\right) \qquad I(x) = I_0\left(1 - 0.5\frac{x}{L}\right)^3 \tag{19}$$

Table 3. First five frequency parameters for uniform EB

Supported	Mode	Exact Solution [28]	Power Series Method [27] (N=25)	N=25		N=50	
				Present Study	Error (%)	Present Study	Error (%)
Simply supported – simply supported (S-S)	1	9.8696	9.8696	9.8761	0.066	9.8712	0.016
	2	39.4784	39.4785	39.5825	0.263	39.5044	0.066
	3	88.8264	87.8912	89.3546	0.591	88.9581	0.148
	4	157.9137	-	159.5884	1.049	158.3300	0.263
	5	246.7401	-	250.8457	1.637	247.7577	0.411
Clamped – simply supported (C-S)	1	15.4182	15.4182	15.4300	0.076	15.4212	0.019
	2	49.9649	49.9623	50.1079	0.285	50.0006	0.071
	3	104.2477	102.3893	104.9033	0.625	104.4112	0.157
	4	178.2697	-	180.2417	1.094	178.76057	0.275
	5	272.0310	-	276.7112	1.691	273.1925	0.425

Table 3. First five frequency parameters for uniform EB (cont.)

Supported	Mode	Exact Solution [28]	Power Series Method [27] (N=25)	N=25		N=50	
				Present Study	Error (%)	Present Study	Error (%)
Clamped – clamped (C-C)	1	22.3733	22.3733	22.3916	0.082	22.3779	0.021
	2	61.6728	61.6611	61.8622	0.306	61.7202	0.077
	3	120.9034	112.0120	121.7020	0.656	121.1028	0.165
	4	199.8594	-	202.1557	1.136	200.4317	0.286
	5	298.5555	-	303.8512	1.743	299.8719	0.439
Clamped – free (C-F)	1	3.5160	3.5160	3.5158	0.006	3.5160	0.000
	2	22.0345	22.0345	22.0540	0.088	22.0394	0.022
	3	61.6972	61.8060	61.8860	0.305	61.7444	0.076
	4	120.9019	-	121.7006	0.656	121.1013	0.165
	5	199.8595	-	202.1558	1.136	200.4317	0.285



a) Mode 1

b) Mode 2

c) Mode 3

d) Mode 4

Figure 2. Convergence of frequencies parameters (φ) of the uniform EB with N

where I_0 and A_0 are characteristics of EB at the left end. Table 4 shows the frequency parameters ($\varphi = \omega\sqrt{\rho A L^4 / E I_0}$) of cantilever tapered EB rested on an uniform Winkler foundation using LCM [29] and this method. The calculated results using B-spline function are fairly close to LCM.

Table 4. Frequency parameters of cantilever tapered EB on uniform foundation (N=30)

K_f	Mode 1		Mode 2		Mode 3	
	LCM [29]	Present study	LCM [29]	Present study	LCM [29]	Present study
0	3.8238	3.8167	18.3173	18.2867	47.2651	47.2513
$5\frac{EI_0}{L^2}$	4.8064	4.8018	18.5214	18.4914	47.3418	47.3281
$50\frac{EI_0}{L^2}$	9.9574	9.9594	20.2721	20.2468	48.0273	48.0146
$100\frac{EI_0}{L^2}$	-	13.5333	-	22.0455	-	48.7668
$500\frac{EI_0}{L^2}$	-	28.7503	-	33.4436	-	54.4391
$1000\frac{EI_0}{L^2}$	39.5692	39.5729	44.2772	44.2988	60.8515	60.8569

Influence of Elastically Restrained Edges on Frequency Parameter Tapered EB on Variable Winkler Foundation

As an interesting application, the influence of variable Winkler foundation and elastically restrained edges on the frequency parameter of tapered EB is evaluated. In this example, the frequency parameter of tapered EB is presented for different values of spring supported at end edges. For this purpose, the tapered EB is considered with general boundary conditions, K_T and K_R. On the other hand, it is considered the tapered EB with linearly varying height and constant width. Also, it is assumed that EB is supported on the variable Winkler foundation (Figure 3). The EB beam characteristics are assumed as follows:

$$A(x)=\begin{cases} A_0\left(1-0.5\frac{x}{L}\right) & x\in\left[0,\frac{L}{2}\right] \\ A_0\left(\frac{1}{2}+0.5\frac{x}{L}\right) & x\in\left[\frac{L}{2},L\right]\end{cases} \qquad I(x)=\begin{cases} I_0\left(1-0.5\frac{x}{L}\right)^3 & x\in\left[0,\frac{L}{2}\right] \\ I_0\left(\frac{1}{2}+0.5\frac{x}{L}\right)^3 & x\in\left[\frac{L}{2},L\right]\end{cases} \tag{20}$$

$$K_f(x)=50\frac{EI_0}{L^2}(1-0.5x) \tag{21}$$

$$K_{TL}=K_{TR}=K_T \qquad K_{RL}=K_{RR}=K_R \tag{22}$$

where A_0 and I_0 are cross-sectional area and second area moment of tarped EB at the right and the left ends. Figure 4 displays 2D contour graph of first frequency parameter of tapered EB on variable Winkler foundation for different spring supporting. Table 5 demonstrates the frequency parameters ($\varphi=\omega\sqrt{\rho AL^4/EI_0}$) of the tapered EB rested on Winkler foundation with linear modulus. It is seen from results that tarped EB on variable foundation can be assumed as clamped beam when values of K_T/EI_0 and K_R/EI_0 are greater than 10000.

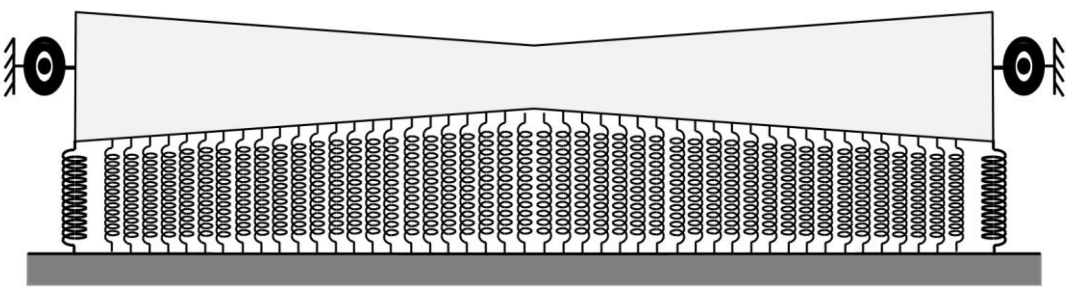


Figure 3. Tapered EB with elastic boundary conditions on variable Winkler foundation

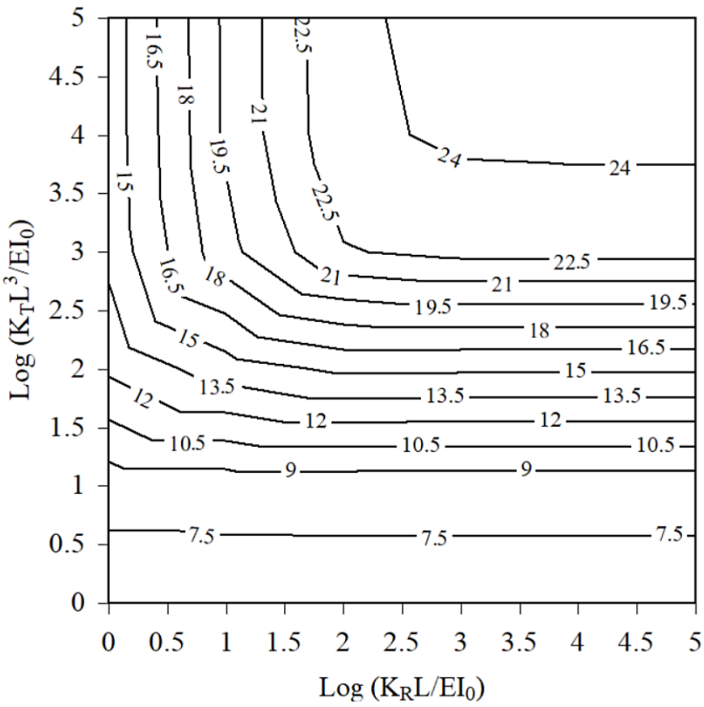


Figure 4. Variation of first frequency parameter of tapered EB on variable Winkler foundation for different spring supporting

Table 5. Frequency parameter for tapered EB with elastic boundary conditions on variable Winkler foundation (N=35)

$K_T L^3/EI_0$	Mode	$K_R L/EI_0$						
		0	1	10	100	1000	10000	100000
0	1	5.6905	6.2348	6.4316	6.4546	6.4569	6.4571	6.4571
	2	7.0497	7.9426	10.1909	10.9964	11.0977	11.1081	11.1091
	3	20.8784	24.4574	33.2389	37.3793	37.9525	38.0121	38.0180
1	1	6.0999	6.4888	6.6318	6.6481	6.6497	6.6499	6.6499
	2	7.3676	8.2772	10.4158	11.1860	11.2830	11.2930	11.2940
	3	21.0554	24.5980	33.3126	37.4298	38.0001	38.0594	38.0653
10	1	8.1181	8.1320	8.1414	8.1427	8.1428	8.1428	8.1428
	2	10.1862	10.8595	12.2345	12.7548	12.8215	12.8284	12.8291
	3	22.6731	25.8746	33.9811	37.8878	38.4322	38.4888	38.4945
100	1	11.4107	12.3026	14.3031	15.3121	15.2414	15.2527	15.2538
	2	22.4807	22.4978	22.5407	22.5606	22.5634	22.5637	22.5637
	3	36.1380	37.2213	40.6551	42.6753	42.9767	43.0084	43.0115
1000	1	12.3567	13.9381	19.0201	22.3865	22.9319	22.9898	22.9957
	2	33.1822	34.6709	40.5439	45.4176	46.2910	46.3853	46.3948
	3	69.5546	70.0591	72.2519	74.2917	74.6755	74.7173	74.7215
10000	1	12.4638	14.1362	19.7325	23.6331	24.2779	24.3466	24.3535
	2	34.7832	36.7397	45.2571	53.4607	55.0463	55.2196	55.2371
	3	79.3806	81.1417	90.5526	102.895	105.735	106.055	106.088
100000	1	12.4746	14.1564	19.8061	23.7617	24.4164	24.4862	24.4932
	2	34.9447	36.9507	45.7556	54.3116	55.9670	56.1478	56.1661
	3	80.3417	82.2578	92.6405	106.440	109.602	109.958	109.994

From results are shown in Table 5, it illustrates that in the tapered EB with $K_T L^3/EI_0 = K_R L/EI_0 = 10000$, 1st, 2nd and 3rd frequency parameter are 24.3466, 55.2196 and 106.055, respectively. In comparison with the tapered EB with $K_T L^3/EI_0 = K_R L/EI_0 = 10000$, maximum difference of 1st, 2nd and 3rd frequency parameter for tapered EB with $K_T L^3/EI_0 = 10000$ and $K_R L/EI_0 = 0$ are approximately 48.81%, 37.01% and 25.15%, respectively. Also, it illustrates that values of 1st frequency parameters of tapered EB on variable Winkler foundation are almost the same when the $K_R L/EI_0$ is larger than 100.

Double Tapered EB on Variable Winkler Foundation

As an interesting application, the influence of the double taper of the beam shape on the frequency parameter is evaluated. In this example, it is assumed that EB is supported on the variable Winkler foundation (Figure 5). The EB beam characteristics are assumed as follows:

$$A(x) = A_0\left(1-\alpha\frac{x}{L}\right)^2 \qquad I(x) = I_0\left(1-\alpha\frac{x}{L}\right)^4 \tag{23}$$

$$K_f(x) = 50\frac{EI_0}{L^2}(1-x) \qquad \xi_e = 0.10 \qquad \xi_i = 0 \qquad L_1 - L_2 = L \tag{24}$$

where I_0 and A_0 are the characteristics of the EB at the left end. Table 6 demonstrates the frequency parameters ($\varphi = \omega\sqrt{\rho A L^4/EI_0}$) of the double tapered EB on Winkler foundation with linear modulus for different values of α.

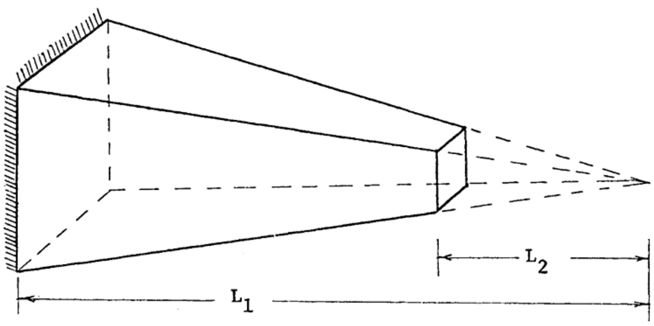


Figure 5. Double tapered EB with elastic boundary conditions on variable Winkler foundation

Table 6. Frequency parameter of tapered EB with elastic boundary conditions on variable Winkler foundation (N=30)

Mode	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
1	4.6556	5.4434	6.7167	9.2514
2	22.3924	21.2963	20.2310	19.5247
3	61.7062	55.3777	48.6353	41.2963
4	120.9467	106.8945	91.6538	74.2248
5	200.5029	176.0969	149.4431	118.4254

CONCLUSION

This paper introduces a numerical solution approach of free vibration analysis of tapered damped EB rested on a variable Winkler foundation. For this proposed, quintic B-spline collocation method is applied to solve governing differential equation. The presented method yields the semi-closed solutions. Therefore, it is more efficient for complex systems. Also, the tapered EB with general boundary conditions can be embedded in this method. The numerical examples are presented to show efficiency and applicability of the presented method. Finally, it efficiency and reliability of the quintic B-spline collocation method are demonstrated from obtained results. Therefore, it can be seen that algorithm converge is increased with the number of terms.

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REFERENCES

[1] A. Ghannadiasl and M. Mofid, "An analytical solution for free vibration of elastically restrained Timoshenko beam on an arbitrary variable Winkler foundation and under axial load," *Latin American Journal of Solids and Structures, an ABCM Journal,* vol. 12, no. 13, pp. 2417-2438, 2015, doi: 10.1590/1679-78251504.

[2] S. Binesh, "Analysis of beam on elastic foundation using the radial point interpolation method," *Scientia Iranica,* vol. 19, no. 3, pp. 403-409, 2012, doi: 10.1016/j.scient.2012.04.003.

[3] N. N. A. Hamid, A. A. Majid, and A. I. M. Ismail, "Quartic B-spline interpolation method for linear two-point boundary value problem," *World Applied Sciences Journal,* vol. 17, pp. 39-43, 2012.

[4] B. Bialecki and G. Fairweather, "Orthogonal spline collocation methods for partial differential equations," *Journal of Computational and Applied Mathematics,* vol. 128, no. 1-2, pp. 55-82, 2001, doi: 10.1016/S0377-0427(00)00509-4.

[5] J. Rashidinia, R. Jalilian, R. Mohammadi, and M. Ghasemi, "Sextic spline method for the solution of a system of obstacle problems," *Applied mathematics and computation,* vol. 190, no. 2, pp. 1669-1674, 2007, doi: 10.1016/j.amc.2007.02.044.

[6] B. Saka and İ. Dağ, "Quartic B-spline collocation method to the numerical solutions of the Burgers' equation," *Chaos, Solitons & Fractals,* vol. 32, no. 3, pp. 1125-1137, 2007, doi: 10.1016/j.chaos.2005.11.037.

[7] M. K. Kadalbajoo and A. S. Yadaw, "B-Spline collocation method for a two-parameter singularly perturbed convection–diffusion boundary value problems," *Applied Mathematics and Computation,* vol. 201, no. 1-2, pp. 504-513, 2008, doi: 10.1016/j.amc.2007.12.038.

[8] M. Ramadan, I. Lashien, and W. Zahra, "Quintic nonpolynomial spline solutions for fourth order two-point boundary value problem," *Communications in Nonlinear Science and Numerical Simulation,* vol. 14, no. 4, pp. 1105-1114, 2009, doi: 10.1016/j.cnsns.2007.12.008.

[9] M.-H. Hsu, "Vibration analysis of non-uniform beams resting on elastic foundations using the spline collocation method," *Tamkang Journal of Science and Engineering,* vol. 12, no. 2, pp. 113-122, 2009, doi: 10.6180/jase.2009.12.2.02.

[10] I. Aziz and B. Šarler, "The numerical solution of second-order boundary-value problems by collocation method with the Haar wavelets," *Mathematical and Computer Modelling,* vol. 52, no. 9-10, pp. 1577-1590, 2010, doi: 10.1016/j.mcm.2010.06.023.

[11] L. B. Da Veiga, C. Lovadina, and A. Reali, "Avoiding shear locking for the Timoshenko beam problem via isogeometric collocation methods," *Computer Methods in Applied Mechanics and Engineering,* vol. 241, pp. 38-51, 2012, doi: 10.1016/j.cma.2012.05.020.

[12] M. Zarebnia and R. Parvaz, "Septic B-spline collocation method for numerical solution of the Kuramoto-Sivashinsky equation," *International Journal of Mathematical and Computational Sciences,* vol. 7, no. 3, pp. 544-548, 2013, doi: 10.5281/zenodo.1088240.

[13] M. Zarebnia and R. Parvaz, "B-spline collocation method for numerical solution of the nonlinear two-point boundary value problems with applications to chemical reactor theory," *International Journal of Mathematical Engineering and Science,* vol. 3, no. 3, pp. 6-10, 2014.

[14] R. Mohammadi, "Sextic B-spline collocation method for solving Euler–Bernoulli Beam Models," *Applied Mathematics and Computation,* vol. 241, pp. 151-166, 2014, doi: 10.1016/j.amc.2014.05.008.

[15] A. Reali and H. Gomez, "An isogeometric collocation approach for Bernoulli–Euler beams and Kirchhoff plates," *Computer Methods in Applied Mechanics and Engineering,* vol. 284, pp. 623-636, 2015, doi: 10.1016/j.cma.2014.10.027.

[16] J. Kiendl, F. Auricchio, L. B. da Veiga, C. Lovadina, and A. Reali, "Isogeometric collocation methods for the Reissner–Mindlin plate problem," *Computer Methods in Applied Mechanics and Engineering,* vol. 284, pp. 489-507, 2015, doi: 10.1016/j.cma.2014.09.011.

[17] G. Akram, "Solution of the system of fifth order boundary value problem using sextic spline," *Journal of the Egyptian Mathematical Society,* vol. 23, no. 2, pp. 406-409, 2015, doi: 10.1016/j.joems.2014.04.009.

[18] F. Auricchio, L. B. da Veiga, J. Kiendl, C. Lovadina, and A. Reali, "Isogeometric collocation mixed methods for rods," *Discrete & Continuous Dynamical Systems-S,* vol. 9, no. 1, p. 33, 2016, doi: 10.3934/dcdss.2016.9.33.

[19] C.-S. Liu and B. Li, "Forced and free vibrations of composite beams solved by an energetic boundary functions collocation method," *Mathematics and Computers in Simulation,* vol. 177, pp. 152-168, 2020, doi: j.matcom.2020.04.020.

[20] Q. Ren and H. Tian, "Numerical solution of the static beam problem by Bernoulli collocation method," *Applied Mathematical Modelling,* vol. 40, no. 21-22, pp. 8886-8897, 2016, doi: 10.1016/j.apm.2016.05.018.

[21] İ. Çelik, "Free vibration of non-uniform Euler–Bernoulli beam under various supporting conditions using Chebyshev wavelet collocation method," *Applied Mathematical Modelling,* vol. 54, pp. 268-280, 2018, doi: 10.1016/j.apm.2017.09.041.

[22] J. Kiendl, F. Auricchio, and A. Reali, "A displacement-free formulation for the Timoshenko beam problem and a corresponding isogeometric collocation approach," *Meccanica,* vol. 53, no. 6, pp. 1403-1413, 2018, doi: 10.1007/s11012-017-0745-7.

[23] W.-R. Chen, "Vibration Analysis of Axially Functionally Graded Tapered Euler-Bernoulli Beams Based on Chebyshev Collocation Method," *International Journal of Acoustics and Vibration,* vol. 25, no. 3, pp. 436-444, 2020, doi: 10.20855/ijav.2020.25.31680.

[24] J. Prochazkova, "Derivative of B-Spline function," in *Proceedings of the 25th Conference on Geometry and Computer Graphics, Prague, Czech Republic*, 2005, pp. 12-16.

[25] M. A. Hilal and H. S. Zibdeh, "Vibration analysis of beams with general boundary conditions traversed by a moving force," *Journal of Sound and Vibration,* vol. 229, no. 2, pp. 377-388, 2000, doi: 10.1006/jsvi.1999.2491.

[26] A. Ghannadiasl, "Analytical Study of Dynamic Response of Railway on Partial Elastic Foundation under Travelling Accelerating Concentrated Load," *International Journal of Transportation Engineering,* vol. 4, no. 4, pp. 317-334, 2017, doi: 10.22119/IJTE.2017.44434.

[27] A. Ghannadiasl, "Natural frequencies of the elastically end restrained non-uniform Timoshenko beam using the power series method," *Mechanics Based Design of Structures and Machines,* vol. 47, no. 2, pp. 201-214, 2019, doi: 10.1080/15397734.2018.1526691.

[28] C. Y. Wang and C. Wang, *Structural vibration: exact solutions for strings, membranes, beams, and plates*. CRC Press, 2019.

[29] A. Ghannadiasl, A. Zamiri, and A. Borhanifar, "Free vibrations of non-uniform beams on a non-uniform Winkler foundation using the Laguerre collocation method," *Journal of the Brazilian Society of Mechanical Sciences and Engineering,* vol. 42, no. 5, pp. 1-12, 2020, doi: 10.1007/s40430-020-02332-3.