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ORIGINAL ARTICLE

Component mode synthesis and stochastic perturbation method for dynamic analysis of large linear finite element with uncertain parameters

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ABSTRACT– In this paper, a method to calculate the first two moments (mean and variance) of the stochastic time response as well as the frequency functions of large FE models with probabilistic uncertainties in the physical parameters is proposed. This method is based on coupling of second order perturbation method and component mode synthesis methods. Various component mode synthesis methods are used to optimally reduce the size of the model. The analysis of dynamic response of stochastic finite element system can be done in the frequency domain using the frequency transfer functions and in the time domain by a direct integration of the equations of motion, using numerical procedures. The statistical first two moments of dynamic response of the reduced system are obtained by the second order perturbation method. Numerical applications have been developed to highlight effectiveness of the method developed to analyze the stochastic response of large structures.

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INTRODUCTION

Due to the large amount of physical variables which can be uncertain, predicting dynamic responses of engineering structures is complex. Models of uncertainty are generally based on either a parametric or non-parametric description of uncertainty. In a parametric description of uncertainty, uncertain variables are described statistically using various techniques such as Monte Carlo Simulation (MCS), the Polynomial Chaos Expansion (PCE) and Perturbation Method (PM). Using (MCS) [1], many samples are generated to run simulations of the dynamic system to obtain the statistics response. For accurate results the sampling number should be large enough, which can take a long time to compute particularly for complex structures with several uncertain properties. Another approach is the PCE method, the response is expanded in terms of a series of polynomials that are orthogonal with respect to mean value operations [2-5]. PCE was first introduced as the homogeneous chaos. Using polynomial chaos expansion (PCE), the stochastic system equations are transformed to a set of deterministic equations. Compared with MCS, the PCE method obtains the statistical characteristics of the results with greatly reduced computational cost. Another alternative approach is the perturbation method (PM), these method based on the Taylor approximation [6], or the Neumann expansion [7], are generally computationally efficient. An improved method, proposed by [8], takes into account the mean and correlation information on uncertain parameters. Recently the perturbation method is used to calculate the natural frequency and frequency response function (FRF) statistics of structures connected to attachments at uncertain locations [9]. Although, depending on applications, they may not give accurate results if the uncertainties are large.

However, as the order of the PM or PCE and the DOF of the dynamic system increases, the number of deterministic equations increases exponentially. To reduce the model order, Component Mode Synthesis (CMS) methods may be employed efficiently in the structural analysis of large-scale structures. In this technique, the global structure is divided in to small substructures, each of which is analyzed independently to obtain its designated solution. These solutions are then assembled to recover the solutions of the global structure by imposing constraints at the interfaces. Depending on the interface condition of the substructures, CMS methods can be classified as fixed interface [10], the free interface [11, 12], or hybrid method [13, 14]. For damping effect, the free interface component mode synthesis methods have used in which the equation of the damped system can be decomposed into a decoupled equation with real coefficient matrices [15]. For large vibroacoustic interaction problems the component mode synthesis method is proposed in [16]. Another approach has been proposed in [17] using experimental data of substructures predicted dynamic behaviour of an assembled structure using the frequency based substructuring. An experimental investigation has been used in [18] to determine the characteristic of fibre metal composite laminates by free vibration analysis.

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Recently [19, 20], used the CMS coupled with polynomial chaos expansions at first and second orders to compute the frequency transfer functions and the time response of stochastic structures. Another coupled approach for interval uncertainty treatment with an improved free interface CMS is developed in [21]. This paper is the continuation of work initiated by [22]. which deals a methodological approach to compute the stochastic eigenmodes of large FE models with parameter uncertainties, this paper examines the effects of uncertainty in the material properties of a beam and plate assembled structure, the model of uncertainty is developed using the perturbation method.

To improve further the computational efficiency, the perturbation method is combined with the CMS sub-structure technique to reduce the model order. In this combined technique, only the order of the finite element model is reduced. By reducing the number of equations in the stochastic model, the computational efficiency is significantly improved while the physical content of the original system is preserved, the analysis of dynamic response of stochastic finite element system can be done in the frequency domain using the frequency transfer functions and in the time domain by a direct integration of the equations of motion, using numerical procedures. Various methods of component mode synthesis to reduce the dimensions of the model are used. The first two moments of dynamic response of structure to a deterministic excitation using a perturbation method are computed.

REDUCTION BY COMPONENT MODE SYNTHESIS

The procedure of the reduction of a large discrete structural system condenses the large number of physical degrees of freedom to fewer generalised coordinates. This so-called reduced order model enables the solution under constrained available computation time, and performs optimization or control applications on a simplified system or extract significant information from the otherwise unwieldy full order model.

The necessity for reduced order models arises because the large number of degrees of freedom involved in a highly refined discretisation still limits the frequent use of full order models, despite the lasting growth of available computation power. At the same time, a coarse discretisation with few degrees of freedom is contradictory to the basic idea of discretisation and usually yields poor results. To avoid the major trade-off between poor results and a too costly computation, a reduction would allow reducing the number of degrees of freedom of the full order system to a smaller number of generalised coordinates in an equivalent reduced system, while retaining the governing characteristics of the high fidelity system.

Model order reduction is implemented using the Component Mode Synthesis (CMS). Techniques are well used for static and dynamic in the analysis of large and complex structures. Let us consider a structure, which is decomposed into n_s substructures $SS^{(k)}$ ($k=1,\ldots, n_s$) which do not overlap.

The equations of motion for each substructures $SS^{(k)}$ are:

$$[M]^{(k)}\{\ddot{y}\}^{(k)}+[C]^{(k)}\{\dot{y}\}^{(k)}+[K]^{(k)}\{y\}^{(k)}=\{f\}^{(k)} \tag{1}$$

For each substructure k the displacement vector $\{y\}^{(k)}$ is partitioned into a vector $\{y_j\}^{(k)}$, called interface dof and $\{y_i\}^{(k)}$ which is the vector of internal dof:

$$\{y\}^{(k)}=\begin{Bmatrix} y_i \\ y_j \end{Bmatrix}^{(k)} \tag{2}$$

The force vector $\{f\}^{(k)}$ is composed into vectors $\{f_j\}^{(k)}$ and $\{f_e\}^{(k)}$, called interface force and external applied force.

$$\{f\}^{(k)}=\{f_j\}^{(k)}+\{f_e\}^{(k)} \tag{3}$$

In the component mode synthesis methods, the physical displacements of the substructure $SS^{(k)}$ are expressed as a linear combination of the substructure modes. After some algebraic transformations, a set of Ritz vectors $[Q]^{(k)}$ is obtained and the displacements of $SS^{(k)}$ are expressed as [6]:

$$\{y\}^{(k)}=[Q]^{(k)}\begin{Bmatrix} y_j^{(k)} \\ \eta_p^{(k)} \end{Bmatrix}=[Q]^{(k)}\{y_c\}^{(k)} \tag{4}$$

where $\{\eta_p\}^{(k)}$ are the generalized coordinates. Details about the used component mode synthesis methods and related matrices $[Q]^{(k)}$ are given in [19]. The conservation of interface dof allows to assemble these matrices as in the ordinary finite element methods. Let us denote by $\mathbf{y}_c$ the vector of independent displacements of the assembled structure:

$$\{y_c\} = \begin{Bmatrix} \eta_p^{(1)} \\ \vdots \\ \eta_p^{(N)} \\ y_j \end{Bmatrix} \tag{5}$$

The compatibility of interface displacements of the assembled structure is obtained by formulating for each substructure $S^{(k)}$ the following relation:

$$\{y_c\}^{(k)} = [\beta]^{(k)} \{y_c\} \tag{6}$$

where $[\beta]^{(k)}$ is the matrix of localization or of geometrical connectivity of the $SS^{(k)}$ substructure, it makes possible to locate the dof of each substructure $SS^{(k)}$ in the global ddl of the assembled structure. They are the Boolean matrices whose elements are 0 or 1.

The free and fixed interface component mode synthesis methods will be used in this paper.A transformation matrix can be defined for each substructure $SS^{(k)}$ by:

$$[Z]^{(k)} = [Q]^{(k)} [\beta]^{(k)} \tag{7}$$

where $[Q]^{(k)}$ is given by the considered CMS method.Thus, the reduced equations of motion can be written as:

$$[M_c]\{\ddot{y}_c\} + [C_c]\{\dot{y}_c\} + [K_c]\{y_c\} = \{f_c\} \tag{8}$$

where the reduced order mass matrix, damping matrix stiffness matrix, displacement vector and force vector are respectively defined as:

$$\begin{aligned} [M_c] &= \sum_{k=1}^{N} {}^t[Z]^{(k)} [M]^{(k)} [Z]^{(k)} \\ [C_c] &= \sum_{k=1}^{N} {}^t[Z]^{(k)} [C]^{(k)} [Z]^{(k)} \\ [K_c] &= \sum_{k=1}^{N} {}^t[Z]^{(k)} [K]^{(k)} [Z]^{(k)} \\ \{y_c\} &= \sum_{k=1}^{N} {}^t[Z]^{(k)} \{y\}^{(k)} \\ \{f_c\} &= \sum_{k=1}^{N} {}^t[Z]^{(k)} \{f\}^{(k)} \end{aligned} \tag{9}$$

STOCHASTIC PERTURBATION METHOD

Stochastic time response

In the stochastic, the basic idea is to expand all random variables and matrices via Taylor series about their spacial expectation using a small parameter. One defines the vector of the average parameters $\overline{\xi}_i$, and the quantity $d\xi_i = \xi_i - \overline{\xi}_i$. All the matrices and vector are random, and are expanded through second order Taylor series as follows:

$$\begin{aligned}
[M_c] &= [M_c]^0 + [M_c]^n \, d\xi_n + [M_c]^{np} \, d\xi_n d\xi_p \\
[C_c] &= [C_c]^0 + [C_c]^n \, d\xi_n + [C_c]^{np} \, d\xi_n d\xi_p \\
[K_c] &= [K_c]^0 + [K_c]^n \, d\xi_n + [K_c]^{np} \, d\xi_n d\xi_p \\
\{f_c\} &= \{f_c\}^0 + \{f_c\}^n \, d\xi_n + \{f_c\}^{np} \, d\xi_n d\xi_p
\end{aligned} \tag{10}$$

where $[\cdot]^0$, $[\cdot]^n$ and $[\cdot]^{np}$ are deterministic matrices corresponding to the zero, the first and the second order partial derivatives with respect to the random parameter and given by:

$$\begin{aligned}
[X]^0 &= \sum_{k=1}^{N} {}^t[Z]^{(k)} \left[X(\xi)\big|_{\bar{\xi}} \right]^{(k)} [Z]^{(k)} \\
[X]^n &= \sum_{k=1}^{N} {}^t[Z]^{(k)} \frac{\partial [X(\xi)]}{\partial \xi_n}\bigg|_{\bar{\xi}} [Z]^{(k)} \\
[X]^{np} &= \sum_{k=1}^{N} {}^t[Z]^{(k)} \frac{\partial [X(\xi)]}{\partial \xi_n \partial \xi_p}\bigg|_{\bar{\xi}} [Z]^{(k)}
\end{aligned} \tag{11}$$

Indicial notations are used with indices n, p running over the sequence 1, 2,….., I as well as the repeated indices summation. The unknown reduced vectors displacement, velocity and acceleration vectors are also developed through Taylor series as follows:

$$\begin{aligned}
\{y_c\} &= \{y_c\}^0 + \{y_c\}^n \, d\xi_n + \{y_c\}^{np} \, d\xi_n d\xi_p \\
\{\dot{y}_c\} &= \{\dot{y}_c\}^0 + \{\dot{y}_c\}^n \, d\xi_n + \{\dot{y}_c\}^{np} \, d\xi_n d\xi_p \\
\{\ddot{y}_c\} &= \{\ddot{y}_c\}^0 + \{\ddot{y}_c\}^n \, d\xi_n + \{\ddot{y}_c\}^{np} \, d\xi_n d\xi_p
\end{aligned} \tag{12}$$

Substituting these developments into Eq. (8) and writing the terms of same order give the following differential systems:

Zero order equation:

$$[M_c]^0 \{\ddot{y}_c\}^0 + [C_c]^0 \{\dot{y}_c\}^0 + [K_c]^0 \{y_c\}^0 = \{f_c\}^0 \tag{13}$$

First Order equation:

$$\begin{aligned}
&[M_c]^0 \{\ddot{y}_c\}^n + [C_c]^0 \{\dot{y}_c\}^n + [K_c]^0 \{y_c\}^n + [M_c]^n \{\ddot{y}_c\}^0 \\
&+ [C_c]^n \{\dot{y}_c\}^0 + [K_c]^n \{y_c\}^0 = \{f_c\}^n
\end{aligned} \tag{14}$$

Second Order equation:

$$\begin{aligned}
&[M_c]^0 \{\ddot{y}_c\}^{np} + [C_c]^0 \{\dot{y}_c\}^{np} + [K_c]^0 \{y_c\}^{np} + [M_c]^{np} \{\ddot{y}_c\}^0 \\
&+ [C_c]^{np} \{\dot{y}_c\}^0 + [K_c]^{np} \{y_c\}^0 + 2[M_c]^n \{\ddot{y}_c\}^p + 2[C_c]^n \{\dot{y}_c\}^p + 2[K_c]^n \{y_c\}^p = \{f_c\}^{np}
\end{aligned} \tag{15}$$

Let the time solution of Eqs. (13), (14) and (15) be required from time 0 to time T. The time span under consideration, T, is subdivided into n time intervals $\Delta t = t / n$, and the numerically approximated solution is obtained at times $0, \Delta t, 2\Delta t, \ldots, t, t + \Delta t$. Assuming that the solutions at times $0, \Delta t, 2\Delta t, \ldots, t$ are known and that the solution at time $t + \Delta t$ is required next. According to the Newmark method, the following assumption is used at time $(t + \Delta t)$:

$$\begin{aligned}\{\ddot{y}_c\}(t+\Delta t) &= (\{y_c\}(t+\Delta t)-\{y_c\}(t))/\alpha(\Delta t)^2 - \{\dot{y}_c\}(t)/\alpha(\Delta t) \\ &\quad -(1-2\alpha)\{\ddot{y}_c\}(t)/(2\alpha) \\ \{\dot{y}_c\}(t+\Delta t) &= \delta(\{y_c\}(t+\Delta t)-\{y_c\}(t))/\alpha(\Delta t) - (1-\delta/\alpha)\{\dot{y}_c\}(t) \\ &\quad -(1-\delta/2\alpha)\{\ddot{y}_c\}(t)\,\Delta t\end{aligned} \tag{16}$$

in which the two parameters, α and δ, verify $\delta \geq 0.5$ and $\alpha \geq (\delta+0.5)/4$ in order to get accurate and stable solution. The following notations are used:

$$a_0 = \frac{1}{\alpha \Delta t^2} \quad a_1 = \frac{\delta}{\alpha \Delta t} \quad a_2 = \frac{1}{\alpha \Delta t} \quad a_3 = \frac{1}{2\alpha} - 1 \quad a_4 = \frac{\delta}{\alpha} - 1 \quad a_5 = \frac{\Delta t}{2}\left(\frac{\delta}{\alpha} - 2\right) \quad a_6 = \Delta t(1-\delta) \quad a_7 = \delta \Delta t$$

Based on these notations the following equations are resulted.

Zero Order equation:

$$[K_{eq}]^0 \{y_c\}^0 (t+\Delta t) = \{f_{eq}\}^0 \tag{17}$$

with:

$$[K_{eq}]^0 = [K_c]^0 + a_0 [M_c]^0 + a_1 [C_c]^0$$

$$\begin{aligned}\{f_{eq}\}^0 &= \{f_c\}^0 (t+\Delta t) + [M_c]^0 [a_0 \{y_c\}^0 (t) + a_2 \{\dot{y}_c\}^0 (t) + a_3 \{\ddot{y}_c\}^0 (t)] \\ &\quad + [C_c]^0 [a_1 \{y_c\}^0 (t) + a_4 \{\dot{y}_c\}^0 (t) + a_5 \{\ddot{y}_c\}^0 (t)]\end{aligned}$$

First order equation:

$$[K_{eq}]^0 \{y_c\}^n (t+\Delta t) = \{f_{eq}\}^n \tag{18}$$

with:

$$\begin{aligned}\{f_{eq}\}^n &= \{f_c\}^n (t+\Delta t) + [M_c]^0 [a_0 \{y_c\}^n (t) + a_2 \{\dot{y}_c\}^n (t) + a_3 \{\ddot{y}_c\}^n (t)] \\ &\quad + [C_c]^0 [a_1 \{y_c\}^n (t) + a_4 \{\dot{y}_c\}^n (t) + a_5 \{\ddot{y}_c\}^n (t)] \\ &\quad - [M_c]^n \{\ddot{y}_c\}^0 (t+\Delta t) - [C_c]^n \{\dot{y}_c\}^0 (t+\Delta t) - [K_c]^n \{y_c\}^0 (t+\Delta t)\end{aligned}$$

The $\{\dot{y}_c\}^0 (t+\Delta t)$ and $\{\ddot{y}_c\}^0 (t+\Delta t)$ vectors are obtained by Eq. (16).

Second order equation:

$$[K_{eq}]^0 \{y_c\}^{np} (t+\Delta t) = \{f_{eq}\}^{np} \tag{19}$$

with:

$$\begin{aligned}\{f_{eq}\}^{np} &= \{f_c\}^{np} (t+\Delta t) + [M_c]^0 [a_0 \{y_c\}^{np} (t) + a_2 \{\dot{y}_c\}^{np} (t) + a_3 \{\ddot{y}_c\}^{np} (t)] \\ &\quad + [C_c]^0 [a_1 \{y_c\}^{np} (t) + a_4 \{\dot{y}_c\}^{np} (t) + a_5 \{\ddot{y}_c\}^{np} (t)] \\ &\quad - [M_c]^{np} \{\ddot{y}_c\}^0 (t+\Delta t) - [C_c]^{np} \{\dot{y}_c\}^0 (t+\Delta t) - [K_c]^{np} \{y_c\}^0 (t+\Delta t) \\ &\quad - 2[M_c]^n \{\ddot{y}_c\}^p (t+\Delta t) - 2[C_c]^n \{\dot{y}_c\}^p (t+\Delta t) - 2[K_c]^n \{y_c\}^p (t+\Delta t)\end{aligned}$$

The solution of the problem is obtained by successively solving of the following algebraic equations:

$$\begin{aligned}\{y_c\}^0(t+\Delta t) &= \left(\left[K_{eq}\right]^0\right)^{-1}\{f_{eq}\}^0\\ \{y_c\}^n(t+\Delta t) &= \left(\left[K_{eq}\right]^0\right)^{-1}\{f_{eq}\}^n\\ \{y_c\}^{np}(t+\Delta t) &= \left(\left[K_{eq}\right]^0\right)^{-1}\{f_{eq}\}^{np}\end{aligned} \tag{20}$$

The derivative of the physical displacements of each substructure are then obtained by:

Zero order:

$$\{y\}^{0(k)}(t+\Delta t) = [Z]^{(k)}\{y_c\}^0(t+\Delta t) \tag{21}$$

First derivative:

$$\{y\}^{n(k)}(t+\Delta t) = [Z]^{(k)}\{y_c\}^n(t+\Delta t) \tag{22}$$

Second derivative:

$$\{y\}^{np(k)}(t+\Delta t) = [Z]^{(k)}\{y_c\}^{np}(t+\Delta t) \tag{23}$$

The mean and the variance values of $y_{t+\Delta t}$ are given by:

$$\begin{aligned}E[\{y\}(t+\Delta t)] &= \{y\}^0(t+\Delta t) + \frac{1}{2}\{y\}^{np}(t+\Delta t)\,\mathrm{cov}(\xi_n,\xi_p)\\ \mathrm{Var}[\{y\}(t+\Delta t)] &= \sum_{n=1}^{N}(\{y\}^n(t+\Delta t))^2\,\mathrm{var}(\xi_n)\end{aligned} \tag{24}$$

Based on these mathematical procedures, the stochastic time response of assembled structures with uncertain parameters can be investigated

Stochastic frequency functions

In the frequency domain and with a deterministic excitation, the equation of motion can be written in the following form:

$$[D](\omega)\{y\} = \{f\} \tag{25}$$

where $[D](\omega)$ is the dynamic stiffness matrix defined by:

$$[D](\omega) = [K] + i\omega[C] - \omega^2[M] \tag{26}$$

In this paper, a hysteretic damping of coefficient η is considered and the dynamic stiffness matrix is rewritten in the form:

$$[D](\omega) = (1+i\eta)[K] - \omega^2[M] \tag{27}$$

In this analysis the matrices $[K]$ and $[M]$ are constant and frequency independent. The transfer matrix $[H]$ is defined by:

$$[D](\omega).[H] = [I] \tag{28}$$

where H(i,j) is the frequency response at the i^{th} node with applied force at the j^{th} node. In order to reduce the computation, the following vector notations are used:

$$[D](\omega).\{H_j\} = \{\delta_j\} \tag{29}$$

where $\{H_j\}$ and $\{\delta_j\}$ are the j^{th} column vectors of $[H]$ and $[I]$. The column vector $\{H_j\}$ corresponds to the concentrated external force is:

$$\{\delta_j\} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \leftarrow j \tag{30}$$

where j corresponds to a dof of the substructure $SS^{(k)}$. Based on the CMS, the reduced equation is given by:

$$\left((1+i\eta)[K_c] - \omega^2[M_c]\right)\{H_c\}_j = \{f_c\} \tag{31}$$

The condensed matrix $[M_c]$, $[K_c]$ and the condensed vector force $\{f_c\}$ are given by (9). Based on the superposition modal method, the vector $\{H_c\}_j$ can be written as:

$$\{H_c(\omega)\}_j = \sum_{m=1}^{M} \frac{^T\{\phi_c\}_m \{f_c\} \{\phi_c\}_m}{(1+i\eta)\lambda_m - \omega^2} \tag{32}$$

where λ_m are the eigenvalues and $\{\phi_c\}_m$ the eigenvectors corresponding to undamped problem $\left([K_c] - \lambda[M_c]\right)\{\phi_c\} = \{0\}$. The column vector $\{H_c\}_j$ is expanded through the second order Taylor series as:

$$\{H_c(\omega)\}_j = \{H_c(\omega)\}_j^0 + \{H_c(\omega)\}_j^n d\xi_n + \{H_c(\omega)\}_j^{np} d\xi_n d\xi_p \tag{33}$$

The eigenvectors and eigenmodes are expanded through the second order Taylor series as :

$$\begin{aligned} \{\phi_c\}_m &= \{\phi_c\}_m^0 + \{\phi_c\}_m^n d\xi_n + \{\phi_c\}_m^{np} d\xi_n d\xi_p \\ \lambda_m &= \lambda_m^0 + \lambda_m^n d\xi_n + \lambda_m^{np} d\xi_n d\xi_p \end{aligned} \tag{34}$$

Substituting the developments of $\{H_c\}_j$, $\{\phi_c\}_m$ and λ_m into Eq. (35), and equating terms of same order, the first two moments of the vector $\{H_c(\omega)\}_j$ are given by the following formulations:

$$[H_c(\omega)]_j^0 = \sum_{m=1}^{M} \frac{^T\{\phi_c\}_m^0 \{f_c\} \{\phi_c\}_m^0}{(1+i\eta)\lambda_m^0 - \omega^2} \tag{35}$$

$$\left[H_c(\omega)\right]_j^n = \sum_{m=1}^{M} \left[\frac{{}^T\{\phi_c\}_m^n \{f_c\}\{\phi_c\}_m^0 + {}^T\{\phi_c\}_m^0 \{f_c\}\{\phi_c\}_m^n}{(1+i\eta)\lambda_m^0 - \omega^2}\right.$$
$$\left. -(1+i\eta)\frac{{}^T\{\phi_c\}_m^0 \{f_c\}\{\phi_c\}_m^0 \lambda_m^n}{\left((1+i\eta)\lambda_m^0 - \omega^2\right)^2}\right]$$

$$\left[H_c(\omega)\right]_j^{np} = \sum_{m=1}^{M} \left[\frac{{}^T\{\phi_c\}_m^{np} \{f_c\}\{\phi_c\}_m^0 + 2\,{}^T\{\phi_c\}_m^n \{f_c\}\{\phi_c\}_m^p + {}^T\{\phi_c\}_m^0 \{f_c\}\{\phi_c\}_m^{np}}{(1+i\eta)\lambda_m^0 - \omega^2}\right.$$
$$-(1+i\eta)\frac{2({}^T\{\phi_c\}_m^n \{f_c\}\{\phi_c\}_m^0 + {}^T\{\phi_c\}_m^0 \{f_c\}\{\phi_c\}_m^n)\lambda_m^p)}{\left((1+i\eta)\lambda_m^0 - \omega^2\right)^2}$$
$$\left. -(1+i\eta)\frac{{}^T\{\phi_c\}_m^0 \{f_c\}\{\phi_c\}_m^0 \lambda_m^{np}}{\left((1+i\eta)\lambda_m^0 - \omega^2\right)^2} + 2(1+i\eta)^2 \frac{{}^T\{\phi_c\}_m^0 \{f_c\}\{\phi_c\}_m^0 \lambda_m^n \lambda_m^p}{\left((1+i\eta)\lambda_m^0 - \omega^2\right)^3}\right]$$

The derivatives of the column vector $\{H(\omega)\}_j$ of the transfer matrix $[H]$ corresponding to the substructure $SS^{(k)}$ where $[Z]^{(k)}$ is assumed to be deterministic are then given by :

$$\{H(\omega)\}_j^{0(k)} = [Z]^{(k)} \{H_c(\omega)\}_j^0$$
$$\{H(\omega)\}_j^{n(k)} = [Z]^{(k)} \{H_c(\omega)\}_j^n \qquad (36)$$
$$\{H(\omega)\}_j^{np(k)} = [Z]^{(k)} \{H_c(\omega)\}_j^{np}$$

Note that the vector $\{H\}$ is complex:

$$\{H\}^0 = \{X\}^0 + j\{Y\}^0; \qquad \{H\}^n = \{X\}^n + j\{Y\}^n \quad ; \qquad \{H\}^{np} = \{X\}^{np} + j\{Y\}^{np} \qquad (37)$$

The frequency response function magnitude is denoted by $MH = \sqrt{X^2 + Y^2}$ and its second order perturbation is given by:

$$MH = MH^0 + MH^n d\xi_n + MH^{np} d\xi_n d\xi_p \qquad (38)$$

where:

$$MH^0 = \sqrt{X^{0^2} + Y^{0^2}}$$
$$MH^n = \frac{X^0 X^n + Y^0 Y^n}{MH^0} \qquad (39)$$
$$MH^{np} = \frac{X^n X^p + Y^n Y^p - MH^n MH^p + X^{np} X^0 + Y^{np} Y^0}{MH^0}$$

The mean and the variance values of the frequency response function magnitude are given by:

$$E[MH] = MH^0 + \frac{1}{2} MH^{np} \operatorname{cov}(\xi_n, \xi_p)$$
$$\operatorname{Var}[MH] = \sum_{n=1}^{N} (MH^n)^2 \operatorname{var}(\xi_n) \qquad (40)$$

These relationships will be used to analyse the dynamic behaviour of assembled structures with uncertain parameters in frequency domain.

NUMERICAL SIMULATION

Stochastic Beam

In order to demonstrate the efficiency of this methodological approach, some benchmark tests are elaborated for beam with linear and nonlinear stochastic parameters. For beam structures, the stochastic frequency response function and the time response of beams with uncertain parameters under stochastic inputs are investigated. The transverse vibration of an Euler beam discretised by 100 simple FE is considered. Each node has 2 dof in-plane rotation and a transverse displacement. The beam is of length L and of circular cross-section with radius r. In order to use the presented CMS methods, the beam is assumed to be composed of two substructures $SS^{(1)}$ and $SS^{(2)}$ as presented in Figure 1.

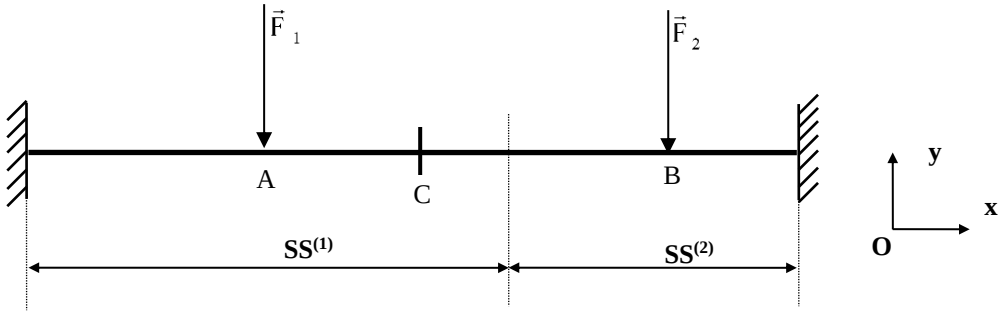

Figure 1. Sub structured clamped beam.

The first substructure consists of 60 finite elements and the second substructure consists of 40. The beam is assumed to be clamped at both ends and the assembled structure has a total of 198 dof. The substructure $SS^{(1)}$ has 120 dof in which 2 are the interface dof and the substructure $SS^{(2)}$ has 80 dof in which 2 are the interface dof. Let E, ρ and η denote element Young modulus, mass density and hysteretic damping coefficient.

The pulsation range of interest is chosen to be 0-2000rd/s. For the Craig-Bampton method (CB) and the free interface method (FI), the substructure modes whose pulsations are smaller than a cut-out pulsation defined by $\omega_{cp} = 2.\omega_u$ are selected. For (CB) method, the size of the reduced system is 17, 9 normal modes are retained for the substructure $SS^{(1)}$, 6 modes for $SS^{(2)}$ and 2 interface dof. For (FI) method, 10 normal modes for the substructures $SS^{(1)}$, 7 modes for $SS^{(2)}$, and 2 interface dof are retained. The size of reduced system is thus 19.

The modal parameters calculated by the present component mode synthesis (CMS) method are compared with those directly calculated using the whole structure. Tables 1 give the eigen modes errors based on the following error criteria:

$$\varepsilon_{\lambda} = 100x\frac{\left|\lambda_c - \lambda_{exact}\right|}{\lambda_{exact}}$$
$$\varepsilon_{\phi} = 100x\frac{\left\|\phi_c - \phi_{exact}\right\|}{\left\|\phi_{exact}\right\|} \quad (41)$$

where λ_{exact} and ϕ_{exact} are obtained by solving the whole FE discretized system. It is clear shown that the eigen modes of the entire structure are accurately obtained using the CMS method, and the (FI) method is more accurate.

Table 1. Comparison of beam eigen modes obtained by the whole structure Craig Bampton and Interface Method for deterministic case.

Mode	Whole structure (rd/s)	Fixed Interface Method (rd/s)	Error Eigen values (%)	Error eigen vectors (%)	Free Interface Method (rd/s)	Error Eigen values (%)	Error eigen vectors(%)
1	81.7611	81.7618	0.0008	0.0537	81.7611	0.0000	0.0002
2	225.3777	225.3852	0.0033	0.1670	225.3777	0.0000	0.0023
3	441.8304	441.8583	0.0063	0.3297	441.8305	0.0000	0.0111
4	730.3683	730.8264	0.0627	1.2793	730.3696	0.0002	0.0629
5	1091.045	1091.403	0.0329	1.2124	1091.048	0.0003	0.1039
6	1523.856	1527.933	0.2675	4.1079	1523.924	0.0044	0.4691
7	2028.805	2037.082	0.4080	5.8452	2029.058	0.0125	0.8887

For stochastic case, let us note that some random parameters such as Young modulus and mass density intervene linearly, and others such the radius intervenes nonlinearly in the stiffness and mass matrices. This nonlinear effect is harder to be analysed. The frequency responses are computed based on the reduced model obtained by CMS methods. The fixed interface method (CB) and the free interface method (FI) are used. The pulsation range is $[0, \omega_u = 2000\,rd/s]$. The substructure modes whose pulsations are smaller than a cut-out pulsation defined by $\omega_{cp} = 2\omega_u = 4000$ rd/s are selected and eleven eigen modes are considered in this study.

For this example, the radius parameter is supposed to be a random variable and defined as follows:

$$r = r_0\left(1 + \frac{\sigma_r}{r_0}\xi_r\right) \tag{42}$$

where ξ_r is a zero mean value Gaussian random variable, $r_0 = 0.01\text{m}$ is the mean value and σ_r is the standard deviation of this parameter. In this nonlinear case, the perturbation method combined with the fixed interface method (CB) and the free interface method (FI) are developed. Firstly, the mean and variance of the magnitude of localized frequency responses H(1,1) have been computed by the proposed approach. The obtained results are compared with those given by the direct Monte Carlo simulation 500 simulations. The obtained results are plotted in Figures 2 and 3 for $\sigma_r = 2\%$.

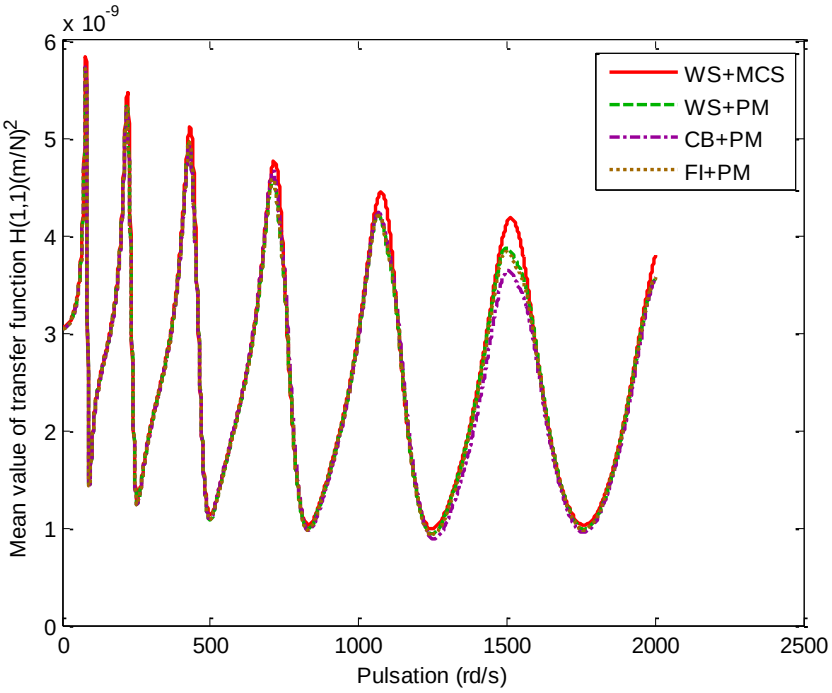

Figure 2. Mean value of transfer function H(1,1), Monte Carlo Simulation (MCS) with 500 samples, Perturbation Method (PM) with Whole Structure (WS) and with Craig Bampton (CB) and Free Interface (FI) methods. $\sigma_r = 2\%$.

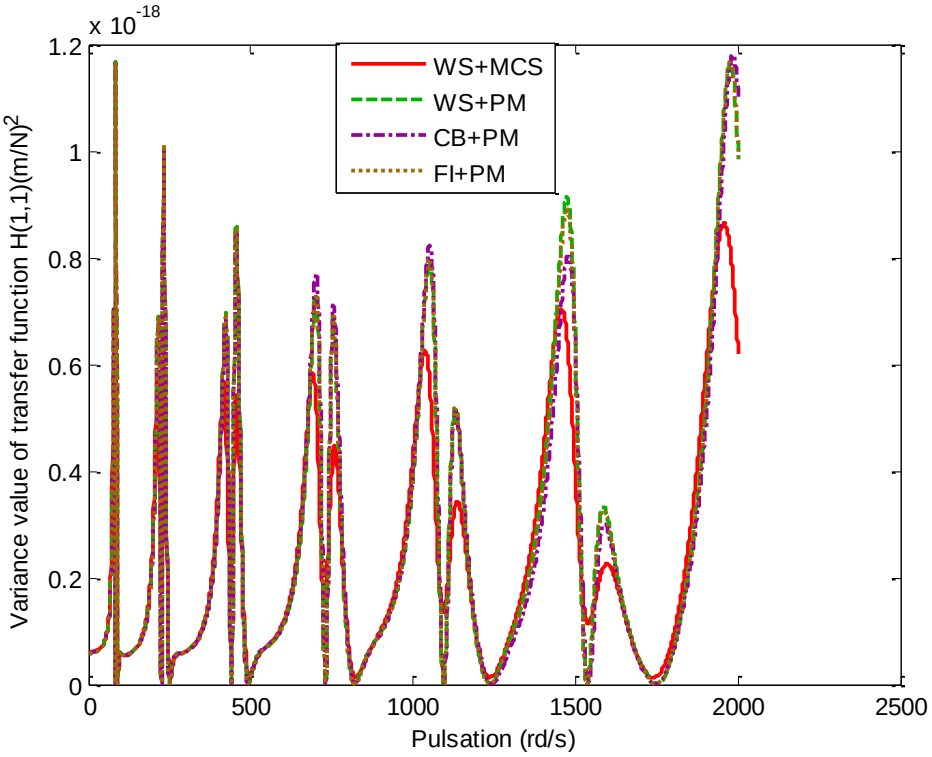

Figure 3. Variance value of transfer function H(1,1), Monte Carlo Simulation (MCS) with 500 samples, Perturbation Method (PM) with Whole Structure (WS) and with Craig Bampton (CB) and Free Interface (FI) methods. $\sigma_r = 2\%$.

These figures show that the obtained solutions provide a very good accuracy as compared with the direct MCS. It was observed for the small pulsations, we obtain the same results, whereas when the pulsation increases the difference between the results increases, as we found in the comparison of structure eigenmodes for deterministic case.

The proposed method with the whole system and the CMS methods require much smaller CPU time than the direct MCS (Table 2). The reduction is the 1,02% for Perturbation method with Whole structure, 0,42% for Perturbation method with Craig Bampton Method and 0,37 for Perturbation method with Free Interface Method.

Table 2. CPU time (s) comparison for stochastic frequency functions of the considered beam, Perturbation method with whole structure and Component Mode Synthesis Methods.

Monte Carlo Simulation with whole structure	Perturbation method with Whole structure	Perturbation method with Craig Bampton Method	Perturbation method with Free Interface Method
158.31	1.63	0.67	0.59

The presented methodological approach is also used to investigate the transient stochastic response of the considered beam with a random radius parameter. Let us consider a structure subjected to vertical loads F_1, and F_2 applied at points **A** and **B** as presented in Figure 1**.**

$$F_1 = f_0 \sin(\omega_2 t)\ ; \qquad F_2 = f_0 \sin(\omega_5 t)\ ; \qquad \text{and} \quad f_0 = 400\ \text{N}$$

where the excitation frequencies ω_i are chosen to be the structural natural frequencies associated to the modes numbers 2 and 5. The obtained results are compared with those obtained by the direct Monte Carlo Simulation (MCS) with 500 samples. Accuracy is observed for the mean value and standard deviation of the vertical displacement at node **C** as clearly shown in Figures 4 and 5. The results did not show much difference because the error observed between the structural natural frequencies associated to the modes numbers 2 and 5 is very small, as we found in the comparison of structure eigenmodes for deterministic case (Table 1).

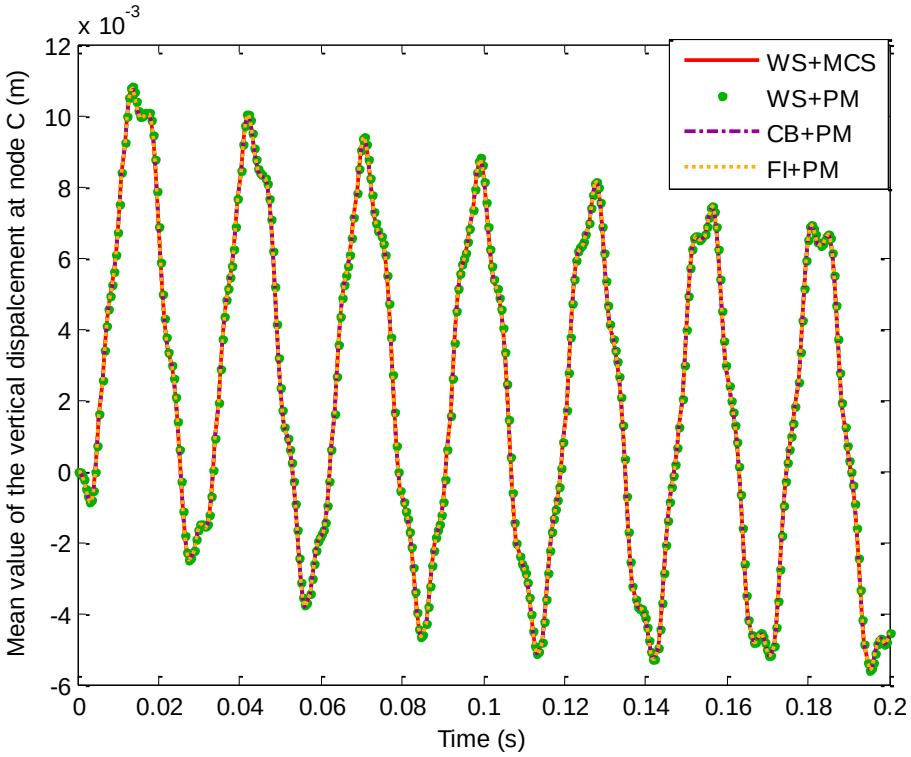

Figure 4. Mean value of the vertical displacement at the node C, Monte Carlo Simulation (MCS) with 500 samples, Perturbation Method (PM) with Whole Structure (WS) and with Craig Bampton (CB) and Free Interface (FI) methods. $\sigma_r = 2\%$.

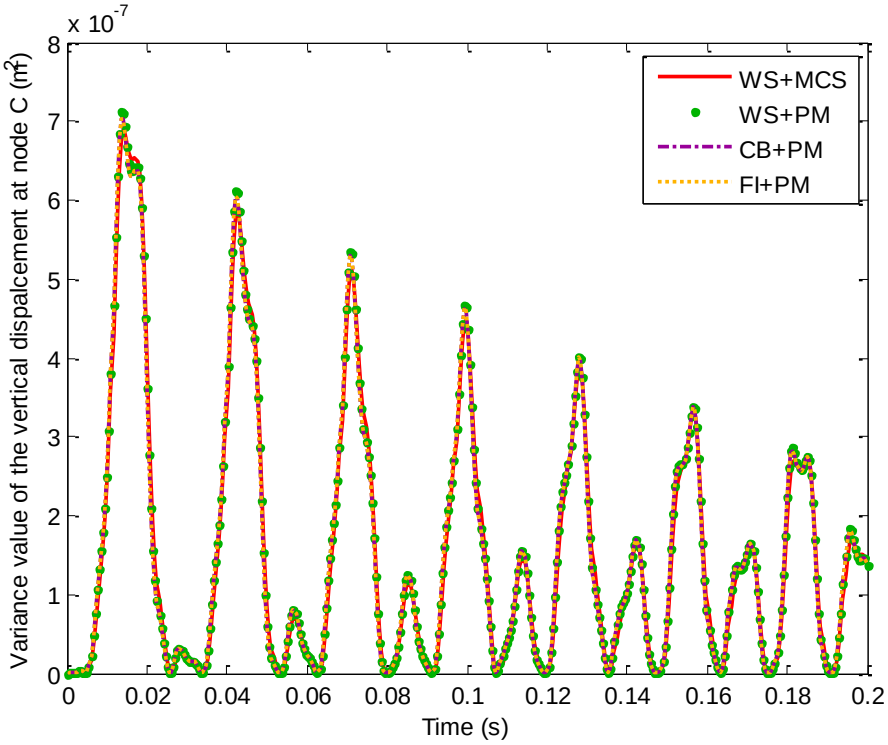

Figure 5. Variance value of the vertical displacement at the node C, Monte Carlo Simulation (MCS) with 500 samples, Perturbation Method (PM) with Whole Structure (WS) and with Craig Bampton (CB) and Free Interface (FI) methods. $\sigma_r = 2\%$.

In order to highlight the performances of the proposed approach in term of computation cost, the CPU time needed for the whole system and for the used CMS methods is given in table 5. A spectacular time reduction is observed in table (3) when the proposed coupling perturbation and CMS methods are used. The reduction is the 0,98% for Perturbation method with Whole structure, 0,68% for Perturbation method with Craig Bampton Method and 0,62 for Perturbation method with Free Interface Method.

Table 3. CPU time (s) comparison for stochastic time response of the considered beam, Perturbation method with whole structure and Component Mode Synthesis Methods.

Monte Carlo Simulation with whole structure	Perturbation method with whole structure	Perturbation method with Craig Bampton Method	Perturbation method with Free Interface Method
307.308521	3.018976	2.108705	1.928031

Assembled plates

Let us consider an assembled plates fixed at some parts and free at the others. The plate and its finite element discretisation are shown in Figure 6. The plate is discretized using 216 quadrilateral thin plate elements and 735 degrees of freedom (3 DOF/node, only the flexural motion is considered).

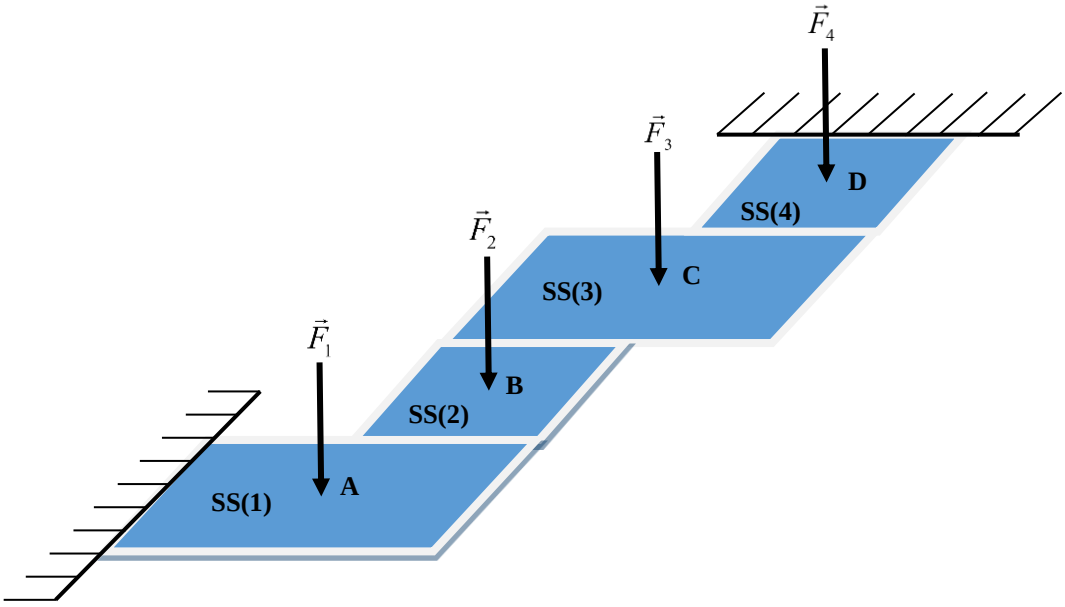

Figure 6. Example of discretisation of structure decomposed in substructures.

In this study the geometrical parameters, the thickness (e = 0.02), the Poisson's ratio ($\nu = 0.3$), and the mass density ρ are assumed to be deterministic $\rho = 7800\,\mathrm{kg/m^3}$. The Young Modulus E is assumed to be independent random variable and defined by:

$$E = E_0\left(1 + \frac{\sigma_E}{E_0}\xi_E\right)$$

where ξ_E are zero mean value Gaussian random variables, $E_0 = 21\times10^{10}\ \mathrm{N/m^2}$ are the mean values of the structural parameters, σ_E are the standard deviation of these parameters. In this study, $\sigma_E = 2\%$ is considered. The structure is assumed to have the hysteretic damping $\eta = 0.005$ that is considered deterministic.

In order to use the coupled perturbationmethod with CMS methods for this example, the structure is divided into four plane plate substructures (see Figure 7). The substructure SS1 has 252 dof in which 21 are the interface dof. The substructure SS2 has 147 dof in which 42 are the interface dof. The substructure SS3 has 273 dof in which 42 are the interface dof. The substructure SS4 has 126 dof in which 21 are the interface dof. The substructure modes whose pulsations are smaller than a cut-out pulsation defined by $\omega_{cp} = 2.\omega_7$, are selected. For a fixed interface method, the size of the reduced system is 81 in which we retain respectively 7 normal modes for substructures SS1 and SS3, 2 modes for SS2 and SS4 and 63 interface dof. For the free interface method, we retain respectively 9 normal modes for substructures

SS1 and SS2, 13 modes for SS3, 6 modes for SS4, 3 rigid body modes for SS2 and SS3 and 63 interface dof. The size of reduced system is thus 106.

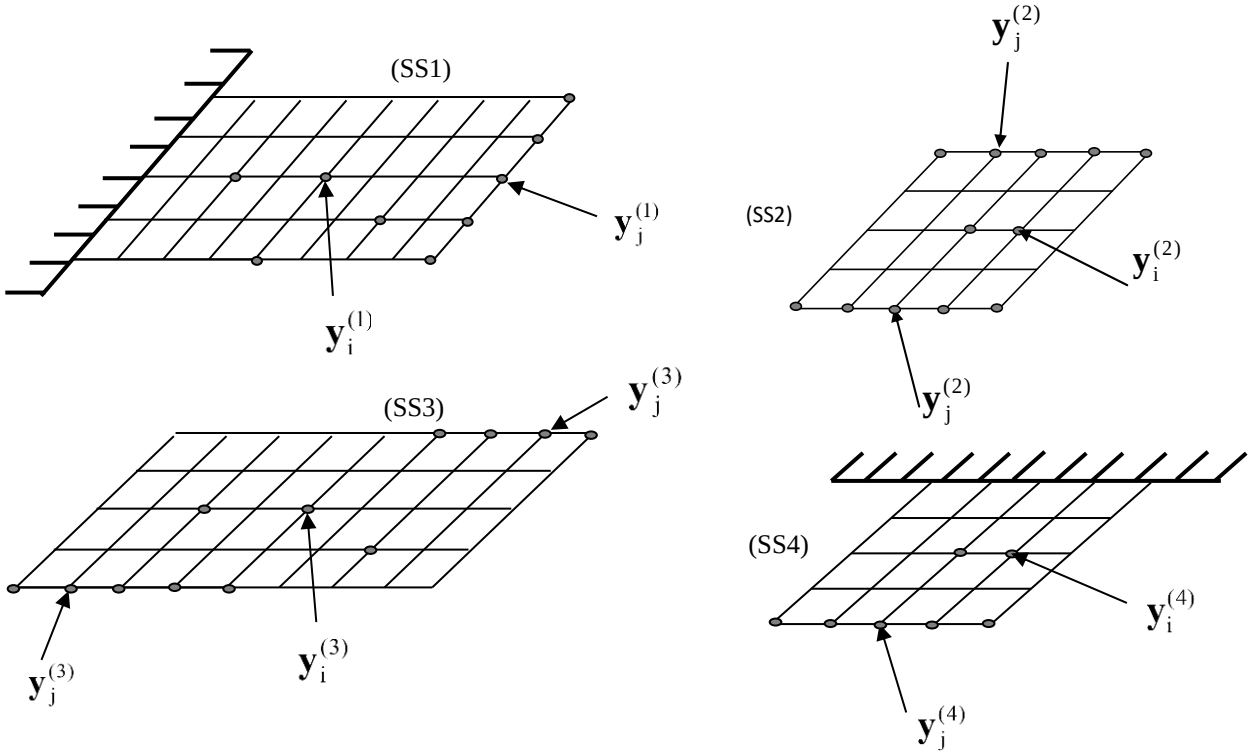

Figure 7. Four substructures (SS1, (SS2), (SS3) and (SS4), interface dof vector $\mathbf{y}_j^{(k)}$ and internal dof vector $\mathbf{y}_i^{(k)}$

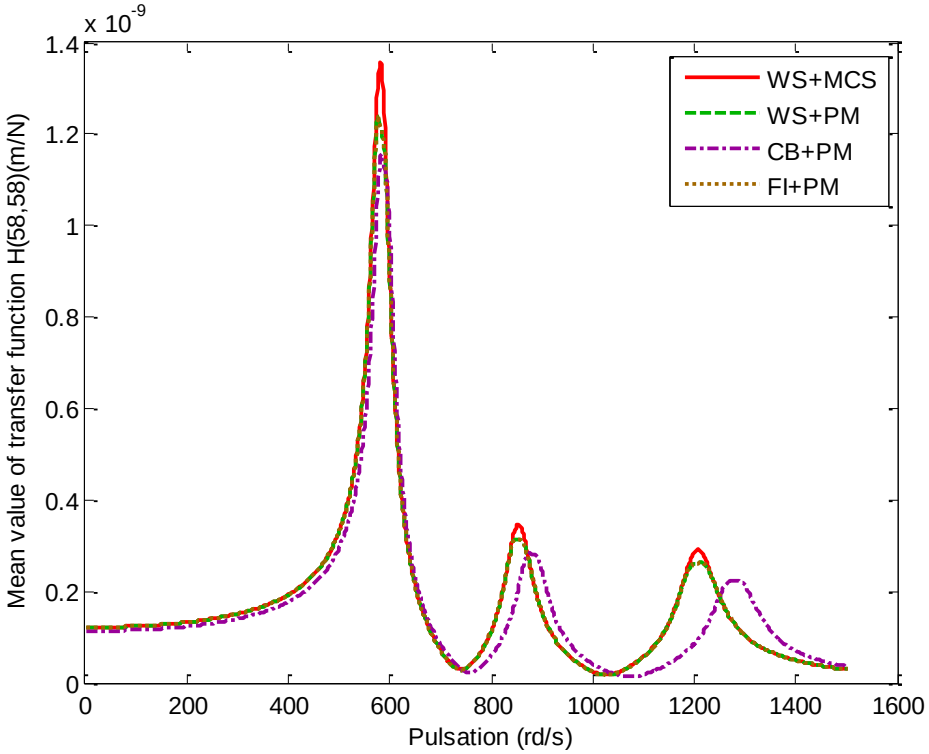

Figure 8. Mean value of transfer function H(58,58), Monte Carlo Simulation (MCS) with 500 samples, Perturbation Method(PM) with Whole Structure(WS) and with Craig Bampton(CB) and Free Interface (FI) methods for $\sigma_E = 2\%$.

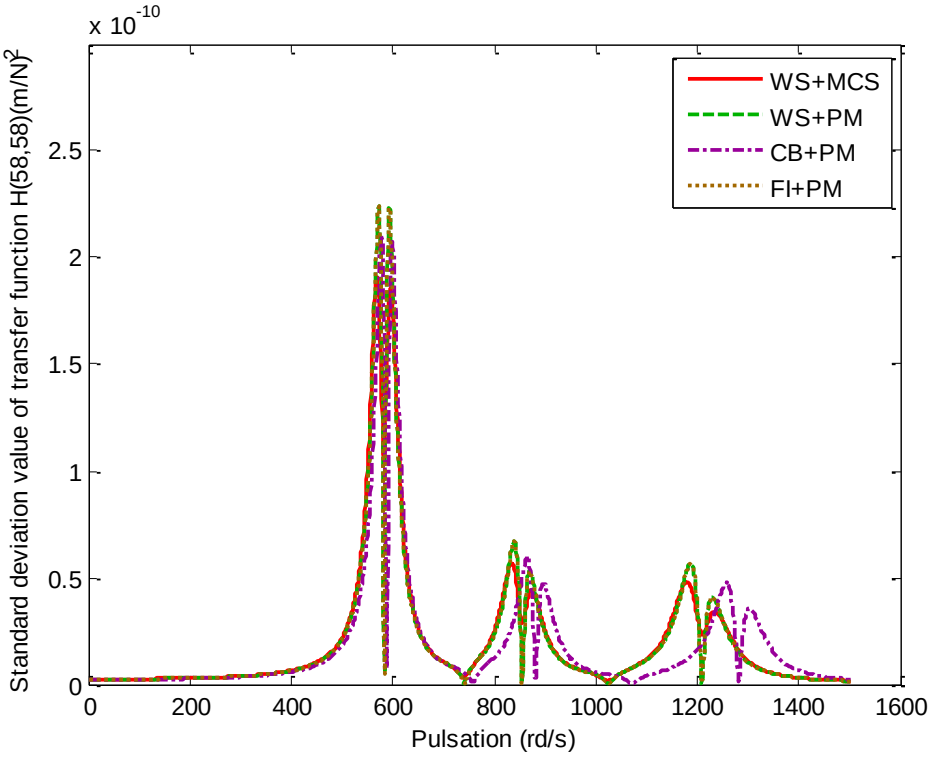

Figure 9. Standard deviation of transfer function H(58,58), Monte Carlo Simulation (MCS) with 500 samples, Perturbation Method (PM) with Whole Structure(WS) and with Craig Bampton(CB) and Free Interface (FI) methods for $\sigma_E = 2\%$.

Firstly, the mean and variance of the magnitude of localized frequency responses H(58,58) have been computed by the proposed approach. The obtained results are compared with those given by the direct Monte Carlo simulation 500 simulations. The obtained results are plotted in Figures 8 and 9 for $\sigma_E = 2\%$.

In the presented numerical analysis, it was observed that the Free Interface Method is more accurate than the Fixed Interface Method (Craig Bampton Method), as we found in the comparison of structure eigenmodes for deterministic case. These figures show that the obtained solutions provide a very good accuracy as compared with the direct MCS. The proposed method with the whole system and the CMS methods require much smaller CPU time than the direct MCS (Table 4).

Table 4. CPU time (s) comparison for stochastic frequency functions of the considered Assembled plates, Perturbation method with whole structure and Component Mode Synthesis Methods.

Monte Carlo Simulation with whole structure	Perturbation method with whole structure	Perturbation method with Craig Bampton Method	Perturbation method with Free Interface Method
1971.95	5.51	2.80	2.32

For time response analysis of structures with uncertain parameters, this plate is subjected to loads F_1 at node **A**, F_2 at node **B** , F_3 at node **C** and , F_4 at node **D** defined by:

$$F_1 = f\sin(\omega_1 t)\ F_2 = f\sin(\omega_3 t)\ F_3 = f\sin(\omega_5 t)\ F_4 = f\sin(\omega_7 t)$$

where ω_j is the structural natural frequency corresponding to the j^{th} mode and the random amlitude f is considered deterministic (f = 200 N).

The time varying mean and variance of the vertical displacement at the node **A** calculated by the developed coupled methods for σ_E = 2% are well compared with those obtained by the direct Monte Carlo simulation with 500 samples. The obtained results are plotted in Figures 10 and 11, and show good accuracy in comparison with each other.

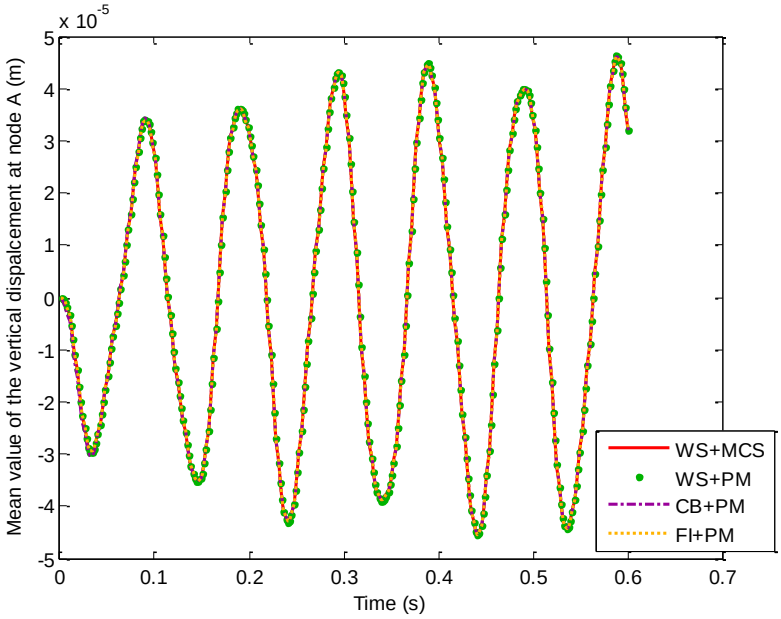

Figure 10. Mean value of the vertical displacement at the node A, Monte Carlo Simulation (MCS) with 500 samples, Perturbation Method (PM) with Whole Structure(WS) and with Craig Bampton(CB) and Free Interface (FI) methods for $\sigma_E = 2\%$.

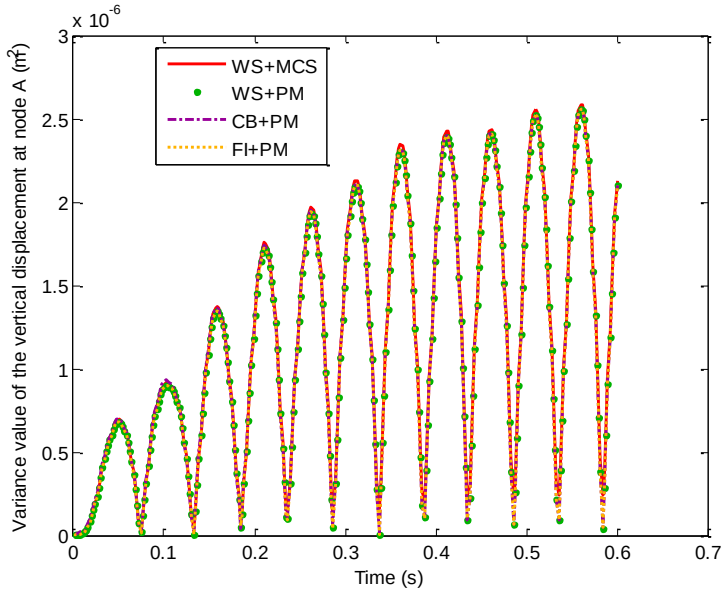

Figure 11. Variance value of the vertical displacement at the node A, Monte Carlo Simulation (MCS) with 500 samples, Perturbation Method (PM) with Whole Structure(WS) and with Craig Bampton(CB) and Free Interface (FI) methods for $\sigma_E = 2\%$.

The CPU time, needed for each proposed approach, is presented in Table 5. It is clearly observed that the developed coupled approaches lead to impressive CPU time reductions.

Table 5. CPU time (s) comparison for stochastic time response of the considered Assembled plates, Perturbation method with whole structure and Component Mode Synthesis Methods.

Monte Carlo Simulation with whole structure	Perturbation method with Whole structure	Perturbation method with Craig Bampton Method	Perturbation method with Free Interface Method
3932.89	13.49	9.65	8.13

CONCLUSIONS

A methodological approach based on a coupling of component mode synthesis methods and perturbation method is developed. This approach is used to investigate the frequency transfer functions and the time response of large linear FE models of beams and assembled plates with stochastic parameters. The random response is expanded along the Taylor series about their spacial expectation using a small parameter, in order to compute the first two moments (mean and variance) of the response. Explicit mathematical developments of the proposed coupling procedures are given. Agreement between results obtained by these methods and by the direct Monte Carlo simulation is demonstrated. The presented approaches are efficient and fast computational ones for the stochastic frequency transfer functions and the time response of structures with uncertain parameters for large structural systems with linear or nonlinear random parameters.

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