

RESEARCH ARTICLE

Numerical analysis of energy harvesting characteristics in axially functionally graded beam structures with different geometric profiles for automotive application

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Abstract - The present study presents a wide-ranging numerical investigation of the energy-harvesting characteristics of an axially functionally graded beam structure containing different geometric profiles. Axially functionally graded materials (FGMs) offer exceptional advantages in energy harvesting applications because their varying material properties in the length direction can be optimised for energy conversion. The present study uses the finite element method (FEM) not only to compare different beam configurations but also to analyse them under dynamic loading conditions. Three different geometric profiles, namely uniform, linear, and parabolic variations along the length direction, are used in this analysis. Each profile consists of a piezo-electric layer near the fixed end of the functionally graded beam, enabling it to serve as an energy harvesting system. The material properties of the beam follow a distribution function that defines the grading profile. To find key performance parameters such as power generation, voltage output and natural frequencies under various excitation conditions, the MATLAB platform is used. Both geometric taper ratios, namely width taper and height taper, are used for the analysis, and their influence on overall performance is studied. The results obtained through the numerical analysis provide valuable insight for the proper design of an energy-harvesting device that can be used in autonomous micro-electromechanical systems, structural health monitoring and wireless sensor networks. The parabolic profile beam exhibits an approximately 115% higher voltage response compared to the uniform beam, indicating significantly enhanced electromechanical performance.

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1. Introduction

The rapid demand for self-sustaining and autonomous electronic systems has significantly increased research in the field of energy harvesting technologies, which capture energy from the surrounding environment [1]. Among several existing mechanisms, vibration-based energy harvesting has been recognised as a promising solution for powering low-power electronic devices, especially in applications where replacing batteries is impractical or costly [2]. This ability to convert mechanical energy into electric power offers a viable solution for micro-electromechanical systems such as structural health monitoring, wireless sensor networks and Internet of Things (IoT) devices [3]. In this case, functionally graded materials (FGMs) play a significant role and are composite materials characterised by variation in material properties [4]. Unlike traditional composite materials, FGMs show smooth transitions in mechanical, thermal, and electrical properties that reduce stress concentrations and interface delamination issues. Due to this uniqueness, FGMs play a superior role in various engineering fields, including aerospace, biomedical implants, and energy conversion systems [5]. Further, combining FGM with piezo-electric materials unveils new possibilities for improving energy conversion efficiency [6]. In this context, axially functionally graded beams offer greater tunability of dynamic characteristics to meet specific operational needs. The variation in different material properties helps optimise frequency response and power output [7]. The design of energy harvesting beams is a critical factor which affects their dynamic behaviour and energy conversion performance. For this, different configurations such as uniform, tapered, exponential, and polynomial profiles, and their performance, such as modal behaviours and stress distributions under dynamic loads, are examined.

Advances in computational mechanics and finite element techniques now allow for a more precise study of the complex interactions in FGMs. The finite element method provides researchers with a powerful tool for investigating how mechanical vibrations, material properties, and electromechanical coupling interact in piezo-electric energy-harvesting systems [8]. Computer simulations enable parametric studies that would be too expensive or take too long to conduct with physical experiments. Researchers have studied nonuniform functionally graded piezo-electric cantilever beams under harmonic excitation using both Euler-Bernoulli and Timoshenko beam theories, as well as finite element methods. These simulation results were compared with existing experimental results. However, this method has limitations because it relies solely on simulations and does not involve new experiments by the authors. It uses idealised conditions that may not fully represent real-world situations [9].

Recent studies have focused on harvesting energy from axially functionally graded piezolaminated non-prismatic beams for automotive applications under both mechanical and thermal loading conditions. Researchers have developed a multiphysics finite-element model to simulate tapered cantilever beams with varying height-to-width taper ratios, while keeping the material-gradient parameters constant. It has been found that adding thermal loading greatly increased both displacement and power output amplitudes. Optimised taper designs reached up to 0.15 W/mm², which is a 650%

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improvement over using only mechanical loading. Voltage generation also decreases with greater tapering in both the height and width directions. One major limitation was that the material gradient parameters ($k = 1$, $n_p = 4$) stayed fixed, which could make it harder to apply these results to other materials and shapes in real-world use [10].

Rotating micro-disks with functionally graded materials in two directions, subjected to both heat and pressure, have been examined using the modified couple-stress theory. The Material properties change with temperature and vary along the thickness and radius. Hamilton's principle and the Ritz method have been used to solve the equations in analyses that include geometric nonlinearity. The model accounts for thermal, centrifugal, and pressure effects, and the corresponding influences of parameters such as temperature, speed, pressure, and material gradation on the disks [11]. A new approach has been examined for vibration energy harvesting, where a bimorph pipe with functionally graded materials in three directions carries steady fluid flow. The material properties change along the width, height, and length directions, and electricity is generated due to the presence of piezo-electric layers on the surfaces. The theoretical results obtained have been verified using finite-element simulations of voltage and power. The effects of material gradients, fluid-structure interaction, load resistance, and geometry on performance have been assessed. It has been found that maximum output power can be achieved at lower resistance due to changes in axial and height gradients. Nevertheless, the system became unstable at higher fluid speeds, which may cause energy harvesting to fail and confine its use in fast-flowing applications [12].

Further studies revealed the axially functionally graded piezo-electric material beams and their effect on the behaviour of vibration due to the variation of properties along the length of the beam in exponential and power law. In that work, the effects of natural frequencies due to changes in volume fraction, slenderness ratio, and different boundary conditions were studied. Specific boundary conditions, such as fixed-free and fixed-fixed, have been considered without accounting for temperature and damping effects, which may limit applicability to more complex real-world structures [13]. To improve the performance of functionally graded piezo-electric energy harvesters with geometric non-uniformity, some Researchers have used the Lichtenberg Algorithm and Artificial Neural Networks. In that study, greater emphasis was placed on the thickness of the piezo-electric layers and the tapering ratio to achieve maximum voltage. Finite element-based analysis within Kirchhoff plate theory has been used to model electromechanical behaviour using four-node elements. The Lichtenberg Algorithm is used to solve problems more efficiently, whereas neural networks are used to train the simulation data to obtain optimal design parameters. However, the approach relies on idealised models and pre-made training datasets and may not readily generalise to a wide range of real-world situations [14]. Further research has been carried out on vibration frequencies of functionally graded beams with axial nonhomogeneity in material properties and point masses. In that study, a precise integration approach and the scaled boundary finite element method have been used. This method employs high-order elements only along the beam length, sets up a first-order matrix equation and dual vector method, and solves it using a precise integration approach to obtain accurate stiffness and mass matrices. While this method predicts natural frequencies well and examines the effects of material gradients, boundary conditions, and mass distribution, it assumes linear elasticity, neglects damping, and employs idealised boundary conditions [15].

Functionally graded plate structures supported on elastic foundations with varying stiffness under different boundary conditions have been studied. A quasi-three-dimensional model has been created that combines nonlocal strain-gradient theory with a cosine-based material gradation. A hybrid hyperbolic-parabolic approach has been used to address shear deformation. The Virtual Work method was implemented to derive the governing equation, which was subsequently solved using the Galerkin approach. In this process, both larger plates and nanoplates were used, and the results were compared with previously reported results. The behaviour was studied in detail under the influence of foundation properties, size, and boundary conditions. However, dynamic and thermal effects were not considered throughout the analysis [16]. In continuation of this, a higher-order shear deformation theory has been used for analysing functionally graded porous (FGP) beams and buckling and free vibration in laminated composite (LC) beams. In this study, the Governing equation was derived using Lagrange's principle, and the resulting equations were solved using Legendre-Ritz functions, accounting for porosity variations and various edge constraints. However, the dynamic effects and thermal effects are neglected in this approach. As a result, it is restricted to a limited set of engineering applications [17]. The stability of functionally graded elliptical pipes carrying fluid has been analysed, with nonuniform flow-velocity profiles considered. In this approach, Timoshenko beam theory is employed, accounting for Coriolis and Centrifugal forces in elliptical geometries. The finite element method with modal analysis was used for the process. In this study, linear material properties and ideal-fluid behaviour were utilised. The corresponding effects were studied on the elliptical cross-sections. As a result, it was limited to other geometries and real-world scenarios [18].

An energy harvester made up of Garolite FR-4 epoxy laminate with variable thickness has been studied. In the theoretical analysis, Euler-Bernoulli beam theory was used, and the Rayleigh-Ritz method was used to solve the equation. It has been observed that the model with variable thickness exhibits a more uniform strain distribution, thereby increasing efficiency above 75% [19]. Both Euler-Bernoulli beam theory and Timoshenko beam theory were used for numerical solution and experimental validation of non-graded beams under bridge vibrations caused by moving vehicles. An ideal material gradation and a simplified structural model were used in the study [20]. Recently, the vibration characteristics of an axially functionally graded non-prismatic beam using Timoshenko beam theory were subjected to axial thermal variation in a humid environment. The beam had a varying thickness along the length direction, with a constant width. The material properties were temperature-dependent and distributed axially according to a power law. The governing equation was derived using Hamilton's principle, and subsequently, the harmonic differential quadrature method was used to solve it. Three thermal conditions were used in the analysis, such as linear, nonlinear, and uniform temperature variation with a constant moisture concentration [21]. Recent advances in vibration energy harvesting and piezo-electric systems

have focused on improving energy conversion efficiency through innovative structural designs, analytical modelling, and experimental validation. A fully coupled model of a quarter-wavelength resonator integrated with a piezo-electric backplate has been developed and experimentally validated its effectiveness for vibro-acoustic energy harvesting [22]. A significant enhancement in the performance of piezo-electric energy harvesters has been demonstrated through synergistic structural design approaches, demonstrating improved power generation capabilities [23]. Similarly, a closed-form analytical solution for energy harvesting has been developed from sandwich beams subjected to impact loading, providing valuable insights into the dynamic response and harvested energy characteristics of such systems [24]. In the field of vibration-based diagnostics, the influence of sensor placement on engine misfire detection using artificial neural networks has been investigated, demonstrating that appropriate sensor positioning significantly improves detection accuracy [25]. These studies collectively highlight the growing importance of advanced modelling techniques, structural optimisation, and intelligent sensing methodologies in enhancing the performance of energy harvesting and vibration monitoring systems.

To establish the practical relevance of the proposed energy-harvesting system, particularly for automotive applications, a detailed analysis of realistic vehicle vibrations is required. It has been observed that automotive vibrations develop primarily for two reasons. Firstly, due to vibration induced by road conditions, which vary from very poor to very good. Secondly, vibration is induced by the engine of the vehicle system. The performance of the proposed profile beam is evaluated under such realistic excitation scenarios to identify power generation potential, frequency strategies, optimal design parameters, etc. For real-world automotive use. In light of this, a detailed comparison between the existing approach and the methodology used in this study is presented in Table 1.

Table 1. Comparison of existing approaches with the present study

Existing Approaches	Present Study	New Knowledge Contribution
Primarily transversely graded FGMs; uniform material distribution along the beam length [26]	Axially functionally graded structures with spatially varying properties	Systematic analysis of axial gradation effects on energy harvesting performance under automotive vibrations
Predominantly uniform rectangular cross-sections; limited geometric variation [27]	Multiple profiles: uniform, linear and parabolic configurations	Comprehensive mapping of geometry-gradation synergy for optimised power density
Idealised harmonic or base excitation; single frequency analysis [28]	Realistic automotive vibration spectra from multiple vehicle zones	Application-specific design guidelines for different vehicle mounting locations
Focus on a single objective (power output or bandwidth) [29]	Multi-objective optimisation: power density, bandwidth, and structural durability	Integrated performance metrics for practical automotive implementation
Separate analysis of material or geometric effects [30]	Coupled finite element analysis integrating both gradation and geometry	Predictive model correlating Axially Functionally graded patterns with geometric profiles for enhanced harvesting efficiency
General vibration energy harvesting; laboratory conditions [31]	Automotive-specific: suspension, engine mount, exhaust system vibrations	Vehicle-ready design specifications and optimal configuration recommendations

Due to the rapid electrification of the automotive industry, there is a growing need for new energy-recovery systems to improve vehicle efficiency and reduce reliance on conventional batteries. One major method is to harvest piezo-electric vibration energy using functionally graded materials that offer better electrical and mechanical properties. Axially functionally graded beam structures, where the material properties vary along the length of the beam, can convert energy more effectively than uniform beams. Although several studies have been carried out for transversely graded piezo-electric energy harvesting, the combined effect of changes in beam shape and axial grading has not been studied in automotive settings. There is limited numerical framework relating different beam shapes, such as linear, parabolic and uniform beams, with axial gradient patterns. The present study addresses these gaps and examines the effect of beam geometry and material grading on output power.

2. Methods and Mathematical Formulation

The mathematical formulation consists of modelling an axially functionally graded beam with different geometries, material non-homogeneity along its length, and an electrical interface for measuring power output. The beam is axially functionally graded, with material properties that change along its length. The mathematical expressions for these functionally graded features for uniform, linear, and parabolic distribution profiles are expressed as:

$$E(x) = E_0 \left[1 + k_e \left(\frac{x}{L} \right) \right] \quad (1)$$

$$\rho(x) = \rho_0 \left[1 + k_{p1} \left(\frac{x}{L} \right) + k_{p2} \left(\frac{x}{L} \right)^2 \right] \quad (2)$$

In which the expressions $E(x)$ and $\rho(x)$ are the Young's modulus and density of the beam material. The terms ke , kp_1 , and kp_2 are dimensionless coefficients that determine the shape of the gradients and have values of unity. The Young's modulus varies linearly, while the density follows a parabolic variation along the length of the beam. Equations for the cross-sectional area of the uniform profile beam are presented in Eq. 3. Similarly, the formulations for the cross-sectional area of the linear-profile beam and the parabolic-profile beam are presented in Eqs 4 and 5.

$$A(x) = A_0 \quad (3)$$

$$A(x) = A_0 \left[\left(1 - \alpha \frac{x}{L}\right) \left(1 - \beta \frac{x}{L}\right) \right] \quad (4)$$

$$A(x) = A_0 \left[\left(1 - \alpha \frac{x}{L}\right)^2 \left(1 - \beta \frac{x}{L}\right)^2 \right] \quad (5)$$

The length of the beam is L , and the transverse cross-sectional area is A_0 near the clamped end. α and β represent the width and height tapers for linear and parabolic profiles, respectively, and their values vary from 0 to 1. The cross-sectional area of the beam is modelled using three different geometric profiles: uniform, linear, and parabolic, as represented in Figure 1.

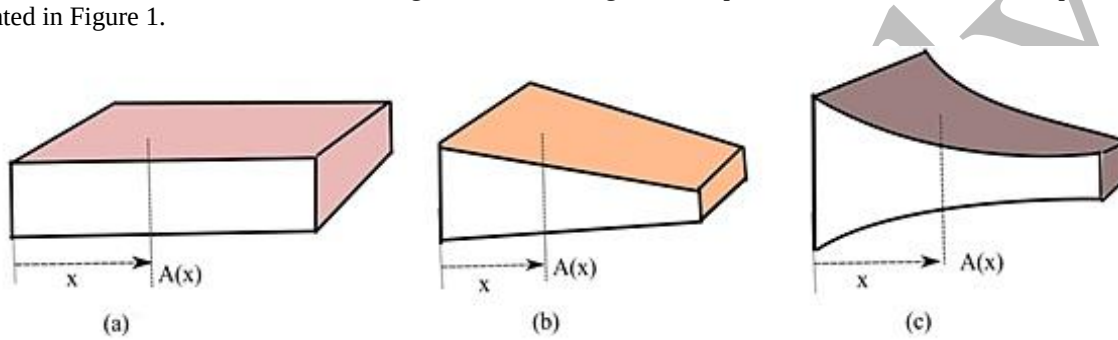


Figure 1. Geometric profiles of the piezo-electric energy harvester: (a) uniform beam, (b) linearly tapered beam, and (c) parabolically tapered beam

The displacement components of the beam along the three mutually perpendicular directions x , y and z can be expressed as follows in Eqs 6.

$$\begin{aligned} u(x, y, z, t) &= -k\theta(x, t) = -k \left(\frac{\partial w}{\partial x} \right) \\ v(x, y, z, t) &= 0 \\ w(x, y, z, t) &= w(x, t) \end{aligned} \quad (6)$$

The beam element is subjected to two degrees of freedom at each node. In finite element modelling, each nodal point is assumed to experience two degrees of freedom, namely transverse displacement (v) and rotation (ψ), which are developed due to the shear force and bending moment. The three-dimensional constitutive equations of piezoelectricity can be represented as in Eqs 7 and 8.

$$\{\sigma\} = [\bar{C}]^E \{\varepsilon\} - [\bar{e}] \{E_e\} \quad (7)$$

$$\{D\} = [\bar{e}]^T \{\varepsilon\} + [\bar{\xi}]^S \{E_e\} \quad (8)$$

where σ , ε are stress and strain tensors, $\bar{C}$ are the compliance matrix coefficients; E_e represents a vector of applied electric field, respectively. The parameter D represents a vector of electric displacement. The terms $\bar{e}$ denote the piezo-electric coupling coefficient; $\bar{\xi}$ represents the permittivity; and $\bar{\xi}$ represents the pyroelectric tensor. Furthermore, since the structure is modelled using Euler–Bernoulli beam theory, all components of stress except the one-dimensional bending stress (σ_1) are considered negligible, thereby modifying the equations and are expressed in Eqs 9 and 10.

$$\sigma_1 = \bar{c}_{11}^E \varepsilon_1 - \bar{e}_{31} E_3 \quad (9)$$

$$D_3 = \bar{e}_{31} \varepsilon_1 + \bar{\xi}_{33}^S E_3 \quad (10)$$

In the present method, for the piezo-electric layer, a single electrical degree of freedom is assumed. The electric potential difference between the top and bottom surfaces of the piezo-electric material is kept constant, while its variation across the transverse direction is linear and is expressed in Eq. 11.

$$-E_{33} = [B_v] \{v\} \quad (11)$$

where B_v is the electric field gradient matrix of the piezo-electric patch, and v is the applied voltage. An axially Functionally Graded cantilever with three different profiles, such as uniform, linear and parabolic profiles, with a piezo-electric patch of length L_p near the clamped end, is shown in Figure 2.

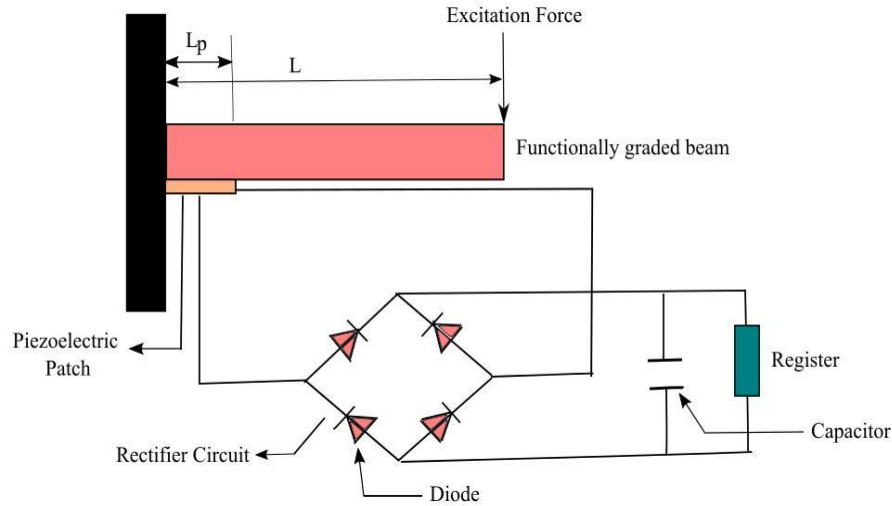


Figure 1. A piezo-laminated axially functionally graded beam having uniform/linear/parabolic profile cross sections with a classic electric interface for vibration energy harvesting

The free end of the proposed beam is subjected to a point load $F(t)$. The dynamic equations of motion for an energy harvesting system can be described using Hamilton's principle as

$$\Pi = \int_{t_1}^{t_2} [\partial(E_k - E_p + W_e) + \delta W] dt = 0 \quad (12)$$

The terms E_k is the kinetic energy, E_p is the total electromechanical enthalpy, and W_e is the work done by the electrical force, δW is the work done due to external mechanical and electrical load. t_1 and t_2 are the time constants at which all first variations vanish. This equation can be used to analyse any mechanical system that comprises piezo-electric patches. The equation of motion of the piezo-laminated, discretised structure is illustrated by

$$\begin{bmatrix} M_b + M_p & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \ddot{v} \end{Bmatrix} + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \dot{v} \end{Bmatrix} + \begin{bmatrix} K_b + K_p & K_{bp} \\ K_{pb} & -K_{pp} \end{bmatrix} \begin{Bmatrix} q \\ v \end{Bmatrix} = \begin{Bmatrix} Q \\ 0 \end{Bmatrix} \quad (13)$$

In addition, the system incorporates supplementary structural damping, which must be accounted for in the analysis. Here, the structural damping is modelled using proportional damping. The stiffness and mass matrices are computed through numerical integration based on the 2-point Gauss quadrature method. The dynamic behaviour of the piezo-thermoelastic beam is determined using the Newmark method. This approach transforms the governing differential equations of motion into a set of algebraic equations as follows.

$$[K_{eff}]^{t+\Delta t} \{q\} = Q_{eff} \quad (14)$$

where K_{eff} and Q_{eff} are the effective stiffness matrix and effective load vector, which can be found as

$$K_{eff} = ([K_b] + [K_p] + [K_{pb}][K_{pp}]^{-1}[K_{bp}]) + a_0([M_b] + [M_p]) + a_1[C] \quad (15)$$

$$Q_{eff} = [M_b + M_p]\{a_0(q_t) + a_2(q_t) + a_3(q_t)\} + [C]\{a_1(q_t) + a_4(q_t) + a_5(q_t)\} \quad (16)$$

The terms $a_0, a_1, a_2, a_3, a_4, a_5$ are represented as

$$\begin{aligned} a_0 &= \frac{1}{\epsilon \Delta t^2}; a_1 = \frac{\delta}{\epsilon \Delta t}; a_2 = \frac{1}{\epsilon \Delta t}; a_3 = \frac{1}{2\epsilon} - 1; \\ a_4 &= \frac{\delta}{\epsilon} - 1; a_5 = \frac{\Delta t}{2} \left(\frac{\delta}{\epsilon} - 2 \right); \epsilon = 0.25; \delta = 0.5. \end{aligned} \quad (17)$$

In order to calculate the output power of the piezo-electric beam, it is presumed that the beam's deformation profile remains unchanged, even though there is a slight increase in bending stiffness due to the piezoelectric layer. The resulting output voltage from the deformation of the PZT patch can be obtained as:

$$[K_{pp}]^{-1}[K_{p\theta}]\{\theta\} + [K_{pp}]^{-1}[K_{pb}]\{q\} \quad (18)$$

For a purely resistive load, the relation between voltage and power can be represented as

$$P_{output} = \frac{v^2}{R_l} \quad (19)$$

3. Results and Discussions

The Finite element formulation has been used to get the results across the three geometric profiles of axially functionally graded beams. This section not only deals with the power-generation capabilities of these profiles but also compares the voltage outputs, natural frequencies and energy-conversion efficiencies. The influence of geometric parameters and material gradient indices on overall harvesting performance has also been studied. For this purpose, the first three nondimensional natural frequencies of the uniform beam were initially calculated as a benchmark using the following equation.

$$\omega_{non-dimensional} = \omega \times L^2 \times \sqrt{\frac{\rho A}{EI}} \quad (20)$$

Table 2 illustrates a comparative sensitivity analysis of the beam using the Euler-Bernoulli and Timoshenko beam theories. The comparison is based on the nondimensional frequencies for the first three modes of vibration, as presented in the table. From the table, it is observed that the deviation between the two theories is very small.

Table 2. First three nondimensional frequencies of the beam using Euler-Bernoulli and Timoshenko beam theories

Parameter	Mode	Euler-Bernoulli Beam theory	Timoshenko Beam theory	Deviation (%)
Nondimensional Frequency	1	3.5160	3.5155	0.0142
Nondimensional Frequency	2	22.0345	22.0312	0.0149
Nondimensional Frequency	3	61.6982	61.6889	0.0150

Similarly, Table 1 presents the first three nondimensional natural frequencies of a uniform beam with one end fixed and the other end free in the energy-harvesting analysis. The fundamental nondimensional frequency is 3.082, corresponding to the primary vibrational mode most relevant for energy-conversion applications.

Table 1. First three nondimensional natural frequencies of a uniform beam with a piezo patch near the fixed end

1st Nondimensional natural frequency	3.082
2nd Nondimensional natural frequency	19.312
3rd Nondimensional natural frequency	54.083

The second and third modes appear at 19.312 and 54.083, respectively, showing the characteristic frequency progression in beam structures. These nondimensional values provide a vital baseline for comparative analysis with other geometric profiles, such as linear and parabolic configurations. The fundamental mode offers maximum energy-harvesting potential due to its optimum displacement amplitude, whereas higher modes of vibration contribute to a broadband frequency response.

Table 2. First nondimensional natural frequency of a linear and parabolic profile beam with variation in width taper (α) = 0.0, 0.3, 0.5, 0.7 and variation in height taper (β) = 0.0, 0.3, 0.5, 0.7

Value of β	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	Linear	Parabolic	Linear	Parabolic	Linear	Parabolic	Linear	Parabolic
$\beta = 0$	3.082	3.082	3.164	3.216	3.213	3.228	3.236	3.072
$\beta = 0.3$	2.690	2.105	2.706	2.030	2.697	1.905	2.649	1.674
$\beta = 0.5$	2.306	1.313	2.278	1.203	2.235	1.086	2.649	1.674
$\beta = 0.7$	1.783	0.624	1.725	0.550	1.663	0.482	1.567	0.395

Table 4 illustrates the first nondimensional natural frequencies for beams with linear and parabolic taper profiles, considering variations in both width taper (α) and height taper (β). To assess the influence of geometric tapering on the beam's dynamic behaviour, the taper values are varied between 0.0 and 0.7. It has been noticed that increasing the width taper (α) by keeping the height of the beam ($\beta = 0$) constant results in a slight increase in natural frequency, which is more noticeable in the linear profile. However, a significant reduction in frequency is observed for both profiles, particularly in the parabolic case as the height taper increases. This is due to the significant decrease in beam stiffness, which reduces the natural frequency. Further, parabolic profiles consistently exhibit lower frequencies than linear ones under the same variation in tapers, due to the curved geometry. The most considerable frequency drop occurs at the highest taper values ($\alpha = 0.7$ and $\beta = 0.7$) in the parabolic beam, which is nearly 0.395 rad/s.

Similarly, **Error! Reference source not found.** represents the second nondimensional natural frequencies of the beam exhibiting parabolic and linear taper profiles. Both the width taper (α) and height taper (β) vary between 0.0 and 0.7. Increasing the width taper (α) and keeping the height taper intact ($\beta = 0$), the natural frequencies with parabolic profiles generally show higher or comparable values than linear ones. However, as the height taper (β) increases, the nondimensional natural frequencies consistently decrease for both beam profiles, more significantly in the parabolic case. For the same taper values, the parabolic profile exhibits sharper frequency reductions than its linear counterpart. Further, it has been observed that for taper levels ($\alpha = 0.7$ and $\beta = 0.7$), the nondimensional natural frequency drops to 2.483 for the parabolic profile and 9.828 for the linear profile. This is due to the greater reduction in structural stiffness, which

results in more flexibility in dynamic behaviour. It is concluded that the height taper has a significant effect on the second mode of vibration frequency compared to the width taper, and the parabolic profile geometry shows a greater amplification on reduction.

Table 5. Second nondimensional Natural frequency of a linear and parabolic profile beam with variation in width taper (α) = 0.0, 0.3, 0.5, 0.7 and variation in height taper (β) = 0.0, 0.3, 0.5, 0.7

Value of β	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	Linear	Parabolic	Linear	Parabolic	Linear	Parabolic	Linear	Parabolic
$\beta = 0$	19.312	19.312	19.825	20.151	20.130	20.225	20.278	19.259
$\beta = 0.3$	16.853	13.189	16.953	12.727	16.897	11.947	16.600	10.507
$\beta = 0.5$	14.445	8.227	14.274	7.544	14.007	6.814	16.600	10.507
$\beta = 0.7$	11.176	3.908	10.818	3.448	10.432	3.028	9.828	2.483

Table 6. Third nondimensional Natural frequency of a linear and parabolic profile beam with variation in width taper (α) = 0.0, 0.3, 0.5, 0.7 and variation in height taper (β) = 0.0, 0.3, 0.5, 0.7

Value of β	$\alpha = 0$		$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$	
	Linear	Parabolic	Linear	Parabolic	Linear	Parabolic	Linear	Parabolic
$\beta = 0$	54.083	54.083	55.508	56.417	56.356	56.626	56.771	53.953
$\beta = 0.3$	47.177	36.918	47.458	35.642	47.306	33.479	46.486	29.473
$\beta = 0.5$	40.438	23.039	39.966	21.145	39.226	19.114	46.486	29.473
$\beta = 0.7$	31.298	10.948	30.306	9.670	29.234	8.499	27.555	6.976

Table 6 summarises the third nondimensional natural frequency of the beams with linear and parabolic taper profiles. Both α and β vary in the range between 0.0 and 0.7, showing a detailed combined geometric effect on higher-mode vibration. From the table, it has been observed that the third nondimensional natural frequency increases gradually as the width taper α increases in the absence of the height taper ($\beta = 0$). However, when β increases, the nondimensional natural frequency decreases for both profiles, and it is more pronounced in the case of parabolic profiles. When both taper values ($\alpha = 0.7$ and $\beta = 0.7$) are considered, the nondimensional natural frequency for the parabolic profile beam is 6.976, whereas for the linear beam it is 27.555. This reduction is due to the substantial loss in stiffness, particularly in a parabolic profile beam subjected to combined tapering. It is observed that the height taper has a greater effect on the third natural frequency than the width taper, and parabolic profile beams are more sensitive to such variations.

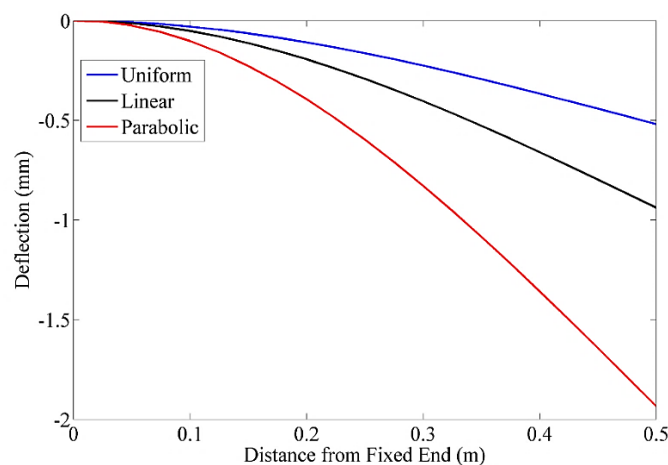


Figure 3. Deflection profiles of cantilever beams with uniform, linear, and parabolic cross-sectional geometries

Figure 3 represents the deflection profiles of three distinct cross-sectional geometries: uniform, linearly tapered, and parabolically tapered configurations. For this purpose, one end of the beam is fixed, and the other end remains free. The figure shows that geometric variation has a significant impact on structural response. It is observed that the tip displacement of the parabolic profile beam is approximately 2.0 mm, which is higher than that of the other profile beams. This is because a considerable reduction in cross-sectional area occurs toward the free end compared to other profiles. This results in a decrease in flexural rigidity and amplifies bending deformation, resulting in the maximum bending moment. In contrast, a uniform profile provides superior resistance to deflection due to constant stiffness along the length of the beam. The linearly tapered profile produces an intermediate deflection of approximately 0.95 mm at the free end, representing a transitional case between the uniform and Parabolic profile beam.

Figure 4 illustrates the bending stress profiles at the top and bottom surfaces of the three cross-sectional geometries: uniform, linearly tapered, and parabolic. For this purpose, the end of the beam is subjected to a point loading, whereas the other end of the beam is fixed. From the figure, it is observed that the parabolic profile beam exhibits maximum tensile

or compressive stress of approximately 10 MPa at the fixed support, indicating maximum material utilisation efficiency. Similarly, the uniform profile beam displays the maximum tensile or compressive stress of approximately 2.5 MPa, whereas in the linearly tapered configuration the value is approximately 5 MPa. From the figure, it is observed that the parabolic profile beam demonstrates superior stress distribution optimisation, resulting in a greater bending moment. Conversely, the uniform cross-section exhibits relatively low stress utilisation near the free end, suggesting material inefficiency. In order to validate the developed MATLAB code, a bimorph cantilever beam composed of two Polyvinylidene fluoride (PVDF) layers is analysed. The beam dimensions are specified as 100 mm in length, 5 mm in width, and 1 mm in thickness. It is discretised into five equal finite elements for numerical analysis. A unit voltage is applied across the thickness, and the resulting deflections are computed at five nodal points.

Table 7. Transverse deflection of a bimorph piezoelectric actuator

Distance(mm) from fixed end	Deflection (μm) [32]	Deflection (μm) [33]	Deflection (μm) present MATLAB code
20	0.0131	0.0138	0.0139
40	0.0545	0.0552	0.0554
60	0.1200	0.1240	0.1247
80	0.2180	0.2210	0.2218
100	0.3400	0.3450	0.3465

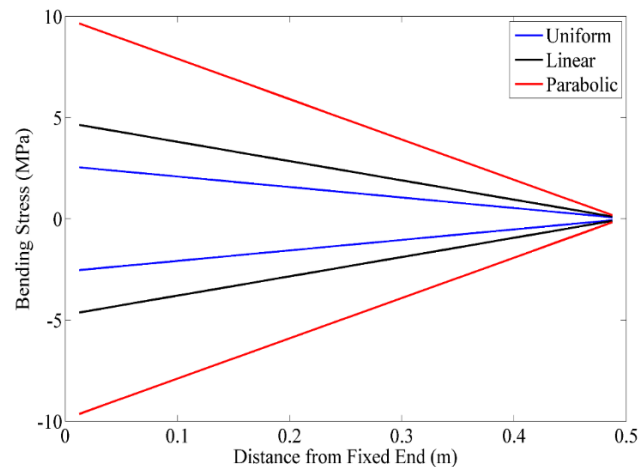


Figure 4. Bending stress distribution in cantilever beams featuring uniform, linearly tapered, and parabolic cross-sections

It is observed from Table 7 that the presently obtained result is in good agreement with the existing results [32] and [33]. The present formulation of the harvester is validated with the available experimental results [34]. For this type of validation, a rectangular cantilever bimorph energy harvester with a tip mass under base excitation is considered, as shown in Figure 5. A similar validation methodology has also been adopted [35].

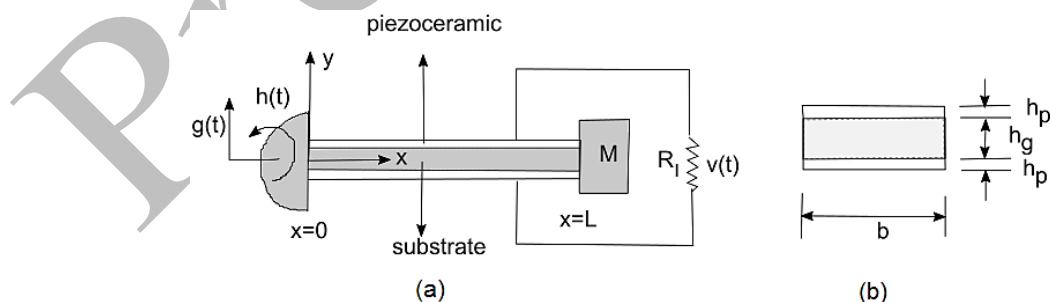


Figure 5. Bimorph cantilever configuration (a) with piezoceramic layers (b) cross-sectional view

Figure 6 (a-b) show the computed results of the voltage and current frequency response function (FRF) using the present formulation. Figure 6 (a-b) show that the present results are in close agreement with the previously published experimental results [24]. A piezo laminated cantilever beam under the simultaneous action of an impulse load at the free end and thermal load has been analysed to study the output responses such as displacement, voltage and output power. The cross-sectional dimensions of the beam near the clamped end have been taken as $b_0 = 0.05$ m and $t_0 = 0.004$ m, respectively. The piezoelectric patch, having a length (L_p) of 0.05 m and a thickness (t_p) of 0.0005 mm, has been attached near the clamped end of the beam of length 0.5 m. The various mechanical, electrical and coupled material properties used near the clamped end are listed in Table 8.

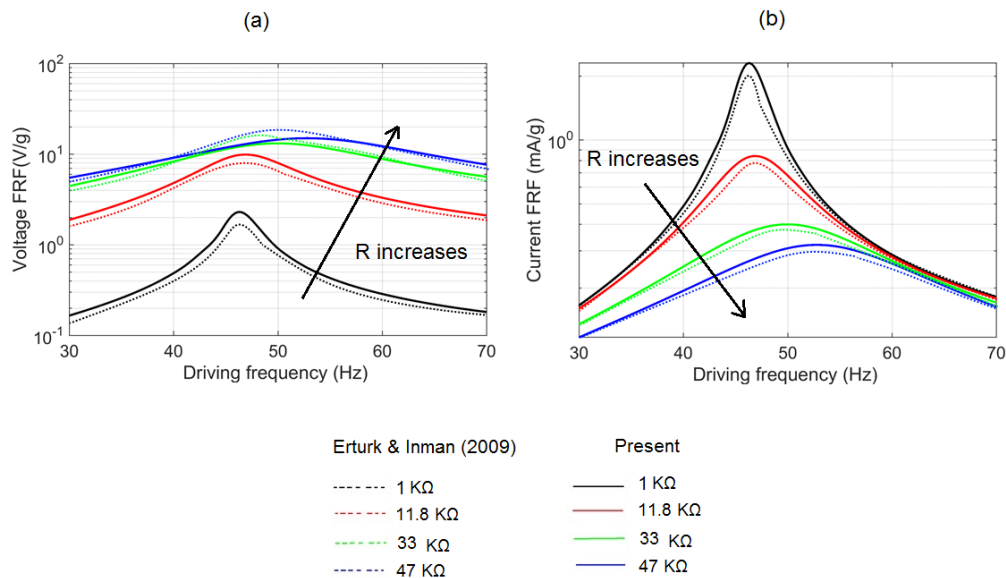
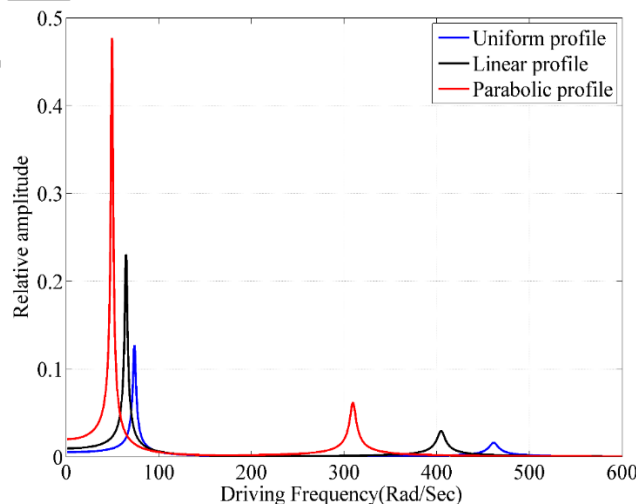
Figure 6. Comparison of voltage FRF for load resistances 1, 11.8 and 33 k Ω

Table 8. Material properties of the beam and PZT

Properties	Steel beam	PZT
E_0 (Gpa)	210	0.0606
μ_0	0.3	0.3
G_0 (Gpa)	80.7	0.0234
ρ_0 (Kg m ⁻³)	7850	7500
e_{31} (C m ⁻²)	-	16.6
ϵ_{33} (nFm ⁻¹)	-	25.55

The following sections represent the frequency domain responses of the proposed beam profiles with variation of width taper (α) and height taper (β). All results presented correspond to axially functionally graded beams incorporating spatially varying material properties as defined in Eq.s (1) and (2). This represents an axial gradient index of unity for the linear variation in modulus and the quadratic variation in density, consistent across all geometric configurations. Thus, the comparisons reflect the combined influence of material gradation and geometric profile (uniform, linear, parabolic) on dynamic response and energy harvesting performance.

Figure 7 displays the frequency-response characteristics of three types of profile beams, such as uniform, linearly tapered and parabolically tapered beams. The system exhibits a frequency variation of up to 600 rad/s. For preliminary parametric investigation, the width taper ($\alpha = 0.2$) and height taper ($\beta = 0.3$) were selected. This is due to the avoidance of stress concentration, numerical instability and excessive stiffness reduction. For Lower values of taper, i.e., $\alpha, \beta < 0.2$, the harvested power and the dynamic response remain close to those of uniform-profile beams, showing limited benefit from geometric modification.

Figure 7. Frequency response of uniform profile beam ($\alpha = 0.0, \beta = 0.0$), linear profile beam ($\alpha = 0.2, \beta = 0.3$) and Parabolic profile beam ($\alpha = 0.2, \beta = 0.3$)

On the contrary, higher taper values, i.e., $\alpha, \beta \geq 0.5$, lead to a pronounced reduction in stiffness and natural frequencies in parabolic profile beams. The intermediate values $\alpha = 0.2$ and $\beta = 0.3$ provide a compromise between stiffness and flexibility, resulting in an increase in harvested power, output voltage and vibration amplitude. Further, it clearly demonstrates the geometry–gradation synergy that efficiently interacts with axial material gradation to concentrate strain energy in the piezoelectric region near the fixed end. Therefore, $\alpha = 0.2$ and $\beta = 0.3$ were selected as representative values to highlight the enhanced energy-harvesting performance and electromechanical coupling without affecting the dynamic stability.

From the figure, it is observed that the parabolic profile system with $\alpha=0.2$ and $\beta=0.3$ exhibits a sharp, high-amplitude resonance peak at approximately 75 rad/s, reaching a relative amplitude of around 0.48. Beyond this resonance frequency, the parabolic profile system shows a secondary, smaller peak at around 310 Rad/Sec before it diminishes at higher frequencies. The parabolic profile system experiences oscillations when excited at its natural frequency. The linear profile system with $\alpha=0.2$ and $\beta=0.3$ has a moderate resonance behaviour, with its primary relative amplitude of 0.23 occurring at around 85 Rad/Sec. The relative amplitude developed is half that of the parabolic profile beam system. This indicates better damping characteristics than a parabolic profile beam system. Further, the linear profile beam system shows a secondary peak at around 400 Rad/Sec followed by a gradual decrease in amplitude. The uniform profile system ($\alpha=0.0$ and $\beta=0.0$) displays the controlled response among the three profile systems. Its primary resonance peak occurs at a similar frequency but has the lowest amplitude, approximately 0.13. Further, the uniform profile system shows additional resonance modes at higher frequencies around 450-500 Rad/Sec, suggesting a more complex dynamic behaviour. All three systems exhibit their primary resonance in the low-frequency range around 100 Rad/Sec. However, their higher-order modal responses and damping properties differ significantly.

Figure 8 represents the voltage response of the proposed beams, such as uniform, linear, and parabolic profile geometries, where the frequency range is from 0 to 600 rad/s. With $\alpha=0.2$ and $\beta=0.3$, the parabolic profile shows the highest resonant response, reaching a voltage amplitude of approximately 280 mV/mm at 75 rad/s. Similarly, the linear profile exhibits moderate resonance of 175 mV/mm at approximately 85 rad/s, and the uniform profile exhibits a maximum of 130 mV/mm at 80 rad/s. These sharp, narrow resonant peaks are due to the highly selective frequency responses with less damping. These voltage-response characteristics are essential for designing and optimising beam-based transducers for selecting appropriate profile geometries. Further, the separation of resonant frequencies also proposes potential for multi-frequency operation using different profile sections. The upward shift in voltage resonance frequency for parabolic profile beams can be attributed not only to the effect of increased Young's Modulus but also to the variation in geometric stiffness toward the tip. This material gradation enhanced the stiffness, sharpening the resonant peak and increasing the voltage output.

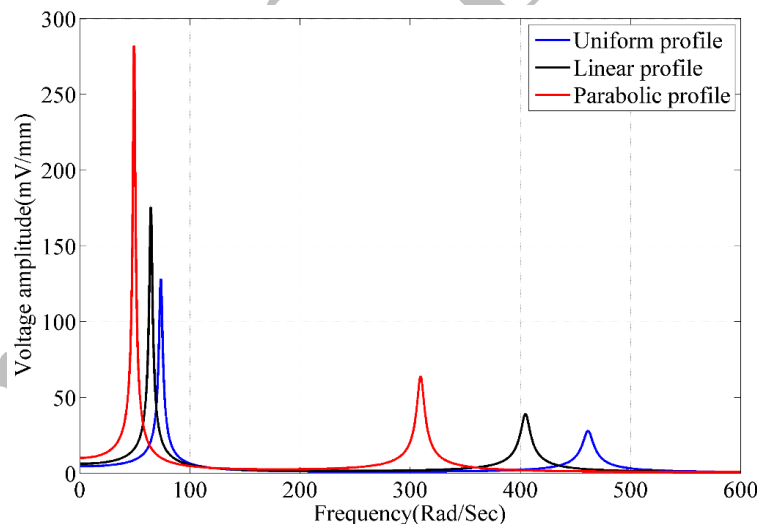


Figure 8. Voltage response of uniform beam ($\alpha = 0.0, \beta = 0.0$), linear profile beam ($\alpha = 0.2, \beta = 0.3$) and Parabolic profile beam ($\alpha = 0.2, \beta = 0.3$)

Figure 9 illustrates the variation of output power response with load resistance for uniform, linear, and parabolic profile beam geometries across a resistance range from 0 to $10 \times 10^5 \Omega$. From the figure, it is evident that the maximum power output for the parabolic-profile beam is approximately $8.3262 \times 10^{-5} \text{ W}$ at an optimal load resistance of approximately 317073 Ω . After this optimal value, the output power gradually decreases as the load resistance increases, in accordance with the classical power output circuit. The linear profile beam shows a power output of approximately $2.5384 \times 10^{-5} \text{ W}$ at the same optimal resistance value. The linear taper profile beam produces a moderate electromechanical coupling efficiency relative to the parabolic profile design. The uniform beam configuration shows the power-generation performance, achieving a maximum output of only $8.69 \times 10^{-6} \text{ W}$, which is lower than the other two profile beams. The uniform cross-section yields less efficient electromechanical energy conversion, although it provides more power delivery across the load conditions. The parabolic profile beam shows an optimised stress distribution along the length of the beam, resulting in improved piezoelectric coupling and uniform strain energy density. This geometric profile optimisation

enables better utilisation of the material volume for power generation. All three designs show maximum power transfer at mid-range resistance values, showing similar internal impedance magnitudes. This indicates that while the power generation capacity varies significantly, the fundamental electrical characteristics remain within comparable ranges. The parabolic density distribution reduces inertia at the free end, resulting in large dynamic strains in the piezoelectric layer, while the gradual increase in Young's modulus ensures mechanical energy transfer.

The dynamic behaviour of three profile beam configurations subjected to time-varying excitation over a period of five seconds is discussed. Figure 10 shows the tip displacement variation of the distinct profile beams subjected to a point load of 1 N at the free end of the beam structure. From the figure, it is observed that the parabolic profile beam exhibits large-amplitude oscillations that reach peak displacements of approximately 7.5×10^{-3} m in the positive direction and -6.5×10^{-3} m in the negative direction during the initial response phase. Afterwards, the response of the beam structure decays exponentially, in which the magnitude of the tip displacement amplitude diminishes gradually. The linear-profile beam exhibits an intermediate dynamic characteristic, with maximum tip displacements of approximately 2.5×10^{-3} m during the initial stage of the response period.

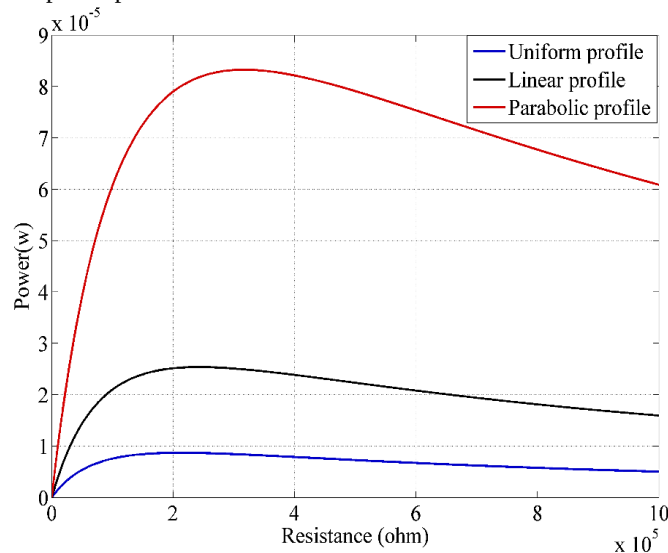


Figure 9. Power response of uniform beam ($\alpha = 0.0, \beta = 0.0$), linear profile beam ($\alpha = 0.2, \beta = 0.3$) and Parabolic profile beam ($\alpha = 0.2, \beta = 0.3$)

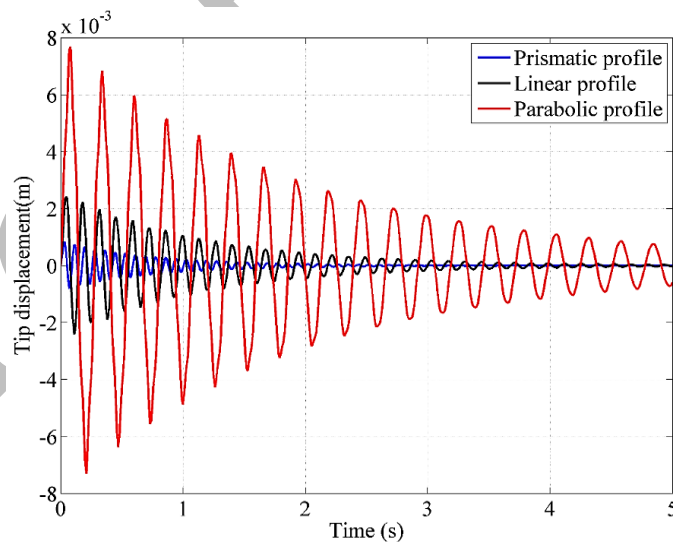


Figure 2. Tip displacement response of uniform beam ($\alpha = 0.0, \beta = 0.0$), linear profile beam ($\alpha = 0.2, \beta = 0.3$) and Parabolic profile beam ($\alpha = 0.2, \beta = 0.3$) in time domain

The frequency appears higher than that of the parabolic profile beam, indicating that linear tapering provides increased structural stiffness for dynamic response. Due to the linear geometry, the decay rate shows moderate damping characteristics. The tip displacement is approximately $\pm 1 \times 10^{-3}$ m for a uniform profile beam at the initial stage of the response period. Further, the amplitude decays rapidly, indicating a stiffer structural system. The extended oscillation period of the parabolic profile beam indicates higher energy-storage capacity and lower structural damping, making it suitable for energy harvesting applications over an extended period. The gradual reduction in amplitude in all three cases reflects natural energy dissipation through material and geometric effects. All three profile configurations show an initial transient behaviour and then a steady-state response, indicating stable system dynamics. The absence of amplitude growth indicates that the excitation frequency remains well-separated from critical natural frequencies for each geometry.

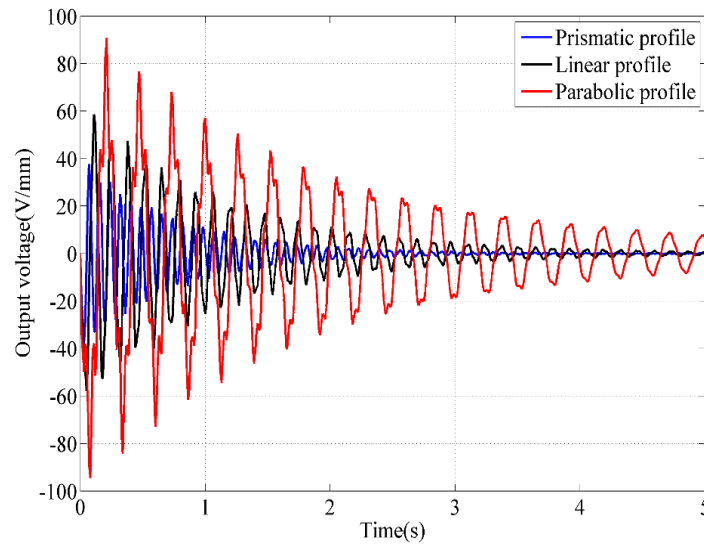


Figure 11. Voltage response of uniform beam ($\alpha = 0.0, \beta = 0.0$), linear profile beam ($\alpha = 0.2, \beta = 0.3$) and Parabolic profile beam ($\alpha = 0.2, \beta = 0.3$) in time domain

Figure 11 demonstrates the voltage-time response characteristics of an energy harvesting system employing three distinct profile geometries over a period of five seconds. In the figure, the ordinate represents the normalised output voltage, whereas the abscissa represents the time period. From the figure, it is evident that the uniform profile beam exhibits a constantly decaying oscillatory response, with initial voltage amplitudes of approximately 40 V/mm. The rapid decrease in voltage amplitude indicates efficient energy dissipation through the harvesting mechanism. The linear profile beam indicates an initial voltage response of 30 V/mm, and subsequently it decays gradually. The parabolic profile beam shows an initial voltage response of 80 V/mm, indicating the maximum energy extraction potential at the beginning of excitation. Afterwards, the response reaches a steady-state condition. The oscillation behaviour shows strong electromechanical coupling, which is used for applications that require instantaneous output power.

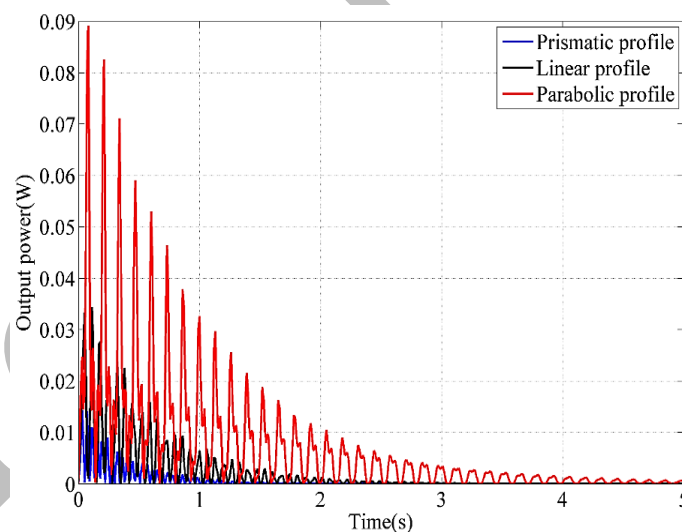


Figure 12. Output Power response of uniform beam ($\alpha = 0.0, \beta = 0.0$), linear profile beam ($\alpha = 0.2, \beta = 0.3$) and Parabolic profile beam ($\alpha = 0.2, \beta = 0.3$) in time domain

Figure 12 shows the output power of an energy harvesting system featuring three different geometric profile configurations for a period of five seconds. Here, the output power (W) represents the ordinate while the load resistance of the circuit represents the abscissa. From the figure, it is observed that, in the initial period, the parabolic profile beam attains a maximum power output of 0.09 W, and subsequently decays as the time increases. This indicates the maximum power captured during the initial stage of oscillation, which is suitable for instantaneous power. The reason for this is the strong electromechanical coupling between the beam structure and the piezoelectric materials. The linear profile shows a power amplitude of 0.03 W, indicating controlled power oscillations compared with the parabolic profile structure. The uniform profile beam structure shows an output power of 0.015 W, which is lower than that of the parabolic and linear profile beam structures.

The differences in performance observed among all these three profiles, such as uniform, linear, and parabolic configurations, cannot be due to the geometry alone. Since all profile beams are axially functionally graded, the variations in mass density and stiffness affect the overall dynamic response of the beam structure. There is a uniform bending

stiffness-to-mass ratio along the uniform beam in linear geometry. As a result, a moderate resonance amplitude develops. In contrast, the parabolic geometry profile improves this effect. The quadratic reduction in cross-sectional area coincides with the linear increase in modulus of elasticity, resulting in concentrated strain energy near the piezoelectric region near the fixed end. This increases the coupling between the material gradation and the geometry to produce high output power. The uniform beam, in the absence of taper variation, shows a lower electromechanical coupling and thus serves as the benchmark for the other two.

4. Conclusions

This study deals with a detailed numerical analysis of the energy-harvesting performance of axially functionally graded beam structures with varying geometric profiles. The finite element method developed in MATLAB was used to compare the natural frequencies, electrical voltage, harvested power, and overall energy conversion efficiency of piezoelectric-integrated beams. The axial functionally graded material properties along the length play a significant role in enhancing electromechanical coupling. Three beam geometries, such as uniform, linearly tapered, and parabolically tapered, are analysed in this paper. The analysis shows that the parabolic-profile beam exhibits higher fundamental frequencies and improved harvesting efficiency. The inclusion of proportional damping and the classic electrical interface improves the dynamic response. It is observed that both tapering ratios and material gradation patterns greatly influence the achievement of maximum voltage and power output. The outcomes of this work provide practical design insights for high-efficiency energy harvesters intended for applications such as wireless sensor networks, structural health monitoring, and self-powered Micro Electro Mechanical System (MEMS) devices.

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Declaration of Conflict of Interest

The authors declare no conflict of interest.

CRedit of Author Contributions

R.R. Senapati: Conceptualization, Methodology
 A. B. Dash: Formal Analysis, Validation, Writing – Original Draft
 S. Acharya: Software, Formal Analysis
 A. R. Biswal: Data Curation, Methodology, Writing Review and Editing
 D. R. Biswal: Supervision, Writing – Review and Editing.

References

- [1] M. Rahmaty, "Energy harvesting technologies in a super-smart city," in *Energy-Efficient Transformative Technologies for Data-Driven Smart Cities*, pp. 55–70, 2026, <https://doi.org/10.1016/b978-0-443-27618-7.00008-x>.
- [2] M. Iqbal, MM Nauman, FU Khan et al., "Vibration-based piezoelectric, electromagnetic, and hybrid energy harvesters for microsystems applications: A contributed review," *International Journal of Energy Research*, vol. 45, no. 1, pp. 65–102, 2021, <https://doi.org/10.1002/er.5643>.
- [3] T. Sanislav, G. D. Mois, S. Zeadally, and S. C. Folea, "Energy harvesting techniques for Internet of Things (IoT)," *IEEE Access*, vol. 9, pp. 39530–39549, 2021. <https://doi.org/10.1109/access.2021.3064066>.
- [4] A. Garg, W. Zheng, R. Raman, and L. Li, "Machine learning in functionally graded materials and nano FGMs: A comprehensive review of predictive modeling for mechanical behavior," *Archives of Computational Methods in Engineering*, vol. 33, no. 1, pp. 533–575, 2026, <https://doi.org/10.1007/s11831-025-10316-6>.
- [5] V. Bhavar, P. Kattire, S. Thakare, and R. K. P. Singh, "A review on functionally gradient materials (FGMs) and their applications," in *IOP Conference Series: Materials Science and Engineering*, vol. 229, no. 1, p. 12021, 2017, <https://doi.org/10.1088/1757-899x/229/1/012021>.
- [6] M. Kumar, M. Suhaib, N. Sharma, S. Kumar, and S. Choudhary, "Energy harvesting technologies in mechanical systems: A comprehensive review," *International Journal of Research Publication and Reviews*, vol. 5, pp. 2782–2787, 2024, <https://doi.org/10.55248/gengpi.5.0124.0303>.
- [7] A. Singh, P. Kumari, and D. Sharma, "Axially functionally graded beams-a review," in *Asian Conference on Mechanics of Functional Materials and Structures*, pp. 479–496, 2022, https://doi.org/10.1007/978-981-99-5919-8_43.
- [8] P. C. Ramegowda and D. Ishihara, "Accurate finite element modelling of multilayered flexible piezoelectric energy harvesting devices with strong coupling of structure, piezoelectricity, and circuit," *International Journal for*

- Computational Methods in Engineering Science and Mechanics*, vol. 25, no. 3, pp. 137–151, 2024, <https://doi.org/10.1080/15502287.2023.2291013>.
- [9] P. Hajheidari, I. Stiharu, and R. Bhat, “Performance of non-uniform functionally graded piezoelectric energy harvester beams,” *Journal of Intelligent Material Systems and Structures*, vol. 31, no. 13, pp. 1604–1616, 2020, <https://doi.org/10.1177/1045389x20930083>.
- [10] A. R. Biswal and D. R. Biswal, “Vibrational energy harvesting from functionally graded nonprismatic beams” *International Journal of Automotive and Mechanical Engineering*, vol. 22, no. 2, pp. 12359–12372, 2025, <https://doi.org/10.15282/ijame.22.2.2025.9.0946>
- [11] S. Pal and D. Das, “Free vibration response of bidirectional functionally graded rotating micro-disk under mechanical and thermal loading,” *The Journal of Strain Analysis for Engineering Design*, vol. 58, no. 7, pp. 517–537, 2023, <https://doi.org/10.1177/03093247231160617>.
- [12] T. Yu, F. Liang, and H. Yang, “Vibration energy harvesting of a three-directional functionally graded pipe conveying fluids,” *Applied Mathematics and Mechanics*, vol. 46, no. 5, pp. 795–812, 2025, <https://doi.org/10.1007/s10483-025-3249-8>.
- [13] B. Gupta, P. Sharma, and S. K. Rathore, “A new numerical modelling of an axially functionally graded piezoelectric beam,” *Journal of Vibration Engineering & Technologies*, vol. 10, no. 8, pp. 3191–3206, 2022, <https://doi.org/10.1007/s42417-022-00550-8>.
- [14] K. Bendine, J. L. J. Pereira, and G. F. Gomes, “Energy harvesting enhancement of nonuniform functionally graded piezoelectric beam using artificial neural networks and Lichtenberg algorithm,” in *Structures*, vol. 57, p. 105271, 2023, <https://doi.org/10.1016/j.istruc.2023.105271>.
- [15] P. Zhang, Z. Peng, H. Xu, and J. Peng, “Modal Characteristics of axially functionally graded beams carrying additional masses,” *Mechanics of Solids*, pp. 1–20, 2025, <https://doi.org/10.1134/s0025654425600904>.
- [16] A. A. Daikh et al., “Static analysis of functionally graded plate structures resting on variable elastic foundation under various boundary conditions,” *Acta Mechanica*, vol. 234, no. 2, pp. 775–806, 2023, <https://doi.org/10.1007/s00707-022-03405-1>.
- [17] N.D. Nguyen, “A new higher-order beam theory for buckling and free vibration responses of laminated composite and functionally graded porous beams,” *The Journal of Strain Analysis for Engineering Design*, vol. 59, no. 1, pp. 67–81, 2024, <https://doi.org/10.1177/03093247231187454>.
- [18] M. Heshmati, F. Daneshmand, and Y. Amini, “Vibration and stability analysis of functionally graded elliptical pipes conveying fluid with flow velocity profile modification,” *Engineering with Computers*, pp. 1–16, 2023, <https://doi.org/10.1007/s00366-021-01541-1>.
- [19] B. Wang, X. Luo, Y. Liu, and Z. Yang, “Thickness-variable composite beams for vibration energy harvesting,” *Composite Structures*, vol. 244, p. 112232, 2020, <https://doi.org/10.1016/j.compstruct.2020.112232>.
- [20] M. Mousavi, S. Ziaei-Rad, and A. H. Karimi, “Piezoelectric-based energy harvesting from bridge vibrations subjected to moving successive vehicles by functionally graded cantilever beams –Theoretical and experimental investigations,” *Mechanical Systems and Signal Processing*, vol. 188, p. 110015, 2023, <https://doi.org/10.1016/j.ymssp.2022.110015>.
- [21] R. Singh and P. Sharma, “Vibration analysis of an axially functionally graded material non-prismatic beam under axial thermal variation in a humid environment,” *Journal of Vibration and Control*, vol. 28, no. 23–24, pp. 3608–3621, 2022, <https://doi.org/10.1177/10775463211037150>.
- [22] M. H. Mansor, M. S. M. Sani, M. F. Hassan, and L. Tang, “Fully-coupled modelling and experimental validation of quarter wavelength resonator with piezoelectric backplate in vibro-acoustic energy harvesting,” *International Journal of Automotive and Mechanical Engineering*, vol. 22, no. 2, pp. 12511–12527, 2025, <https://doi.org/10.15282/ijame.22.2.2025.19.0956>
- [23] J. Zhang, Y. Wang, X. Liu, “Significantly enhanced energy harvesting performance in piezoelectric energy harvesters through synergistic structural design,” *Materials Horizons*, vol. 12, no. 10, pp. 3494–3504, 2025, <https://doi.org/10.1039/d4mh01902d>.
- [24] S. Dao, Nguyen, D. Nguyen, N. Nguyen, D.Thai, “A closed-form solution for harvesting energy from the vibrations of sandwich beams under impact loading,” *Buildings*, vol. 15, no. 12, p. 2135, 2025, <https://doi.org/10.3390/buildings15122135>.
- [25] M. H. Abdelati, A. H. Matar, M. Mourad, and M. Rabie, “Evaluating sensor placement in vibration-based engine misfire detection using artificial neural networks,” *International Journal of Automotive and Mechanical Engineering*, vol. 22, no. 3, pp. 12603–12613, 2025. <https://doi.org/10.15282/ijame.22.3.2025.5.0962>.
- [26] MK. Khorramabadi Free vibration of functionally graded beams with piezoelectric layers subjected to axial load. *Journal of Solid Mechanics*, vol. 1, no. 1, pp. 22–28, 2009, <https://doi.org/10.1007/s10483-009-0803-7>.
- [27] A. Benjeddou, S.R. Mahmoud, M.A. Eltaher, “Active vibration control of functionally graded beams with piezoelectric layers based on higher-order shear deformation theory,” *Composite Structures*, vol. 152, pp. 391–404, 2016, <https://doi.org/10.1007/s11803-016-0352-y>.
- [28] Z. G. Ruan, Z.-G. Ying, H. Lei, L. Xia, and W. Wang, “Base-excited random vibration of quasi-periodic controllable viscoelastic sandwich plates with distributed supporting masses,” *Mechanics of Advanced Materials and Structures*, vol. 33, no. 1, p. 2681941, 2026, <https://doi.org/10.1080/15376494.2026.2681941>.
- [29] A. Abdelkefi “Aeroelastic energy harvesting: A review,” *International Journal of Engineering Science*, vol. 100, pp. 112–135, 2016, <https://doi.org/10.1016/j.ijengsci.2015.10.006>.

- [30] F. Tornabene, N. Fantuzzi, M. Baccocchi, “Free vibration analysis of functionally graded structures using advanced finite element formulations,” *Composite Structures*, vol. 132, pp. 385–410, 2015.
- [31] A. Erturk, D. J. Inman, *Piezoelectric Energy Harvesting*. Chichester: John Wiley & Sons; 2011, <https://doi.org/10.1002/9781119991151>.
- [32] L. Guo and Y. Qin, “Modelling and analysis of a bimorph piezoelectric cantilever,” in *2016 13th International Conference on Ubiquitous Robots and Ambient Intelligence (URAI)*, pp. 248–251, 2016, <https://doi.org/10.1109/urai.2016.7625748>.
- [33] A. Khan, M. Q. Nawaz, and L. Xu, “Investigation and numerical simulation of different piezoelectric bimorph cantilever designs for energy harvesting,” *International Journal of Electrical, Energy and Power System Engineering*, vol. 7, no. 2, pp. 85–99, 2024, <https://doi.org/10.31258/ijeepe.7.2.85-99>.
- [34] A. Erturk and D. J. Inman, “An experimentally validated bimorph cantilever model for piezoelectric energy harvesting from base excitations,” *Smart Materials and Structures*, vol. 18, p. 025009, 2009, <https://doi.org/10.1088/0964-1726/18/2/025009>.
- [35] M. A. Elgamal, H. Elgamal, and S. A. Kouritem, “Optimised multi-frequency nonlinear broadband piezoelectric energy harvester designs,” *Scientific Reports*, vol. 14, no. 1, p. 11401, 2024. <https://doi.org/10.1038/s41598-024-61355-1>.

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