

RESEARCH ARTICLE

Variance reduction through rerandomization and covariate adjustment in factorial design using simulation

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Abstract: This paper develops a unified framework for improving statistical efficiency in factorial experiments by integrating rerandomization and regression-based covariate adjustment. We derive new theoretical results showing that the efficiency gains from these approaches are complementary and multiplicative, driven by the acceptance probability and the predictive power of covariates. We further characterize the covariance structure of factorial effect estimators under constrained randomization and provide comprehensive simulation evidence demonstrating substantial improvements in precision. The results offer new insights into combining design-based and model-assisted strategies for multi-factor experimental settings.

Article History

Received : 10 June 2026
Revised : 15 July 2026
Accepted : 7 September 2026
Published : 30 September 2026

Keywords

Factorial design
Pure randomization
Covariate adjustment
Rerandomization
Relative Efficiency

1. Introduction

Factorial experiments are central to modern experimental design because they enable simultaneous investigation of multiple factors and their interactions on a response variable (1, 2). In a typical 2^k factorial design, experimental units are randomly assigned to combinations of factor levels that allows efficient estimation of both main and interaction effects under relatively weak assumptions [3]. A key principle underlying such designs is randomization, which ensures unbiased estimation of causal effects and provides a valid foundation for statistical inference [4]. Despite its theoretical appeal, complete randomization may produce substantial imbalance in pre-treatment covariates across treatment combinations, particularly in finite samples. Such imbalances can inflate the variance of treatment effect estimators, reduce statistical precision, and potentially obscure true factor effects [5]; [6]. These challenges are more pronounced in factorial settings, where multiple contrasts are estimated simultaneously and covariate imbalance may differentially affect each effect. Rerandomization has emerged as a principled design-based approach to improving covariate balance. It was originally initiated by Morgan [6], the method involves repeatedly randomizing treatment assignments until a pre-specified balance criterion commonly based on the Mahalanobis distance between covariate means is satisfied. By restricting attention to well-balanced allocations, rerandomization enhances covariate balance while preserving the unbiasedness of treatment effect estimators and reducing their variance [6], [7]. Although this method was well studied in treatment control settings, its extension to factorial designs introduces additional complexity due to the multiplicity of treatment contrasts and the need for joint balance across all factor combinations [8]. A complementary strategy for improving efficiency is regression-based covariate adjustment. Methods such as analysis of covariance (ANCOVA) incorporate pre-treatment covariates directly into the analysis, thereby reducing residual variation and improving estimator precision [9], [10]. When applied after randomization, regression adjustment can further mitigate any residual imbalance that persists even under rerandomization.

Rerandomization has become an important design-based tool for improving covariate balance and estimator precision in randomized experiments. The foundational work of Morgan et al. [11] established the theoretical basis for rerandomization in treatment control settings. It demonstrates that restricting the randomization space using a Mahalanobis balance criterion preserves unbiasedness while reducing variance. Subsequent work by [12], [13] further quantified the relationship between acceptance probability and efficiency gains through the variance reduction factor v_a . More recent contributions have extended rerandomization to multi-levels and factorial settings. In particular, [8], [13], [14] developed rerandomization procedures tailored to factorial experiments. Their work emphasized joint covariate balance across multiple treatment contrasts. Similarly, Zhao et al. [13] provided a unified asymptotic framework for rerandomization in general experimental designs that includes factorial structures. In their studies, they established large-sample properties of estimators under constrained randomization.

Since the seminal contributions of [6] and [5], research on rerandomization has expanded substantially. Recent work has generalized rerandomization to multi-arm and factorial experiments, introduced alternative balance criteria based on p-values from covariate balance tests [15], and established asymptotic theory for flexible regression adjustment, doubly robust estimation, and machine-learning estimators under rerandomization [16]. Extensions have also been developed for split-plot experiments [17], high-dimensional LASSO adjustment [18], and settings with missing baseline covariates [19]. Furthermore, [20] demonstrated that rerandomization can mitigate false discoveries arising from post hoc covariate selection, strengthening the theoretical justification for rerandomization beyond efficiency gains alone. Collectively, these studies illustrate that modern rerandomization has evolved from a simple covariate-balancing technique into a

comprehensive design framework that supports efficient estimation, robust inference, and flexible analysis across increasingly complex experimental settings.

Despite these advances, the existing literature has largely focused on design stage improvements through rerandomization, with comparatively less attention to the interaction between rerandomization and analysis stage adjustments, such as regression or ANCOVA. While regression adjustment has been studied extensively in randomized experiments. For instance, a study by Lin [9], explained that joint effect with rerandomization particularly in factorial designs remains underdeveloped both theoretically and empirically. Hence, this study contributes to existing literature along three main dimensions:

First, we provide a unified theoretical framework that explicitly links rerandomization and regression-based covariate adjustment in factorial experiments. By deriving the joint variance expression, we show that the efficiency gains from these two strategies are complementary and multiplicative, that operates through distinct mechanisms in that rerandomization reduces covariate imbalance via the acceptance probability p_a , while regression adjustment removes the explainable component of outcome variation through R_X^2 . Secondly, this study extends existing rerandomization theory by deriving the covariance structure of factorial effect estimators under constrained randomization. While prior work has established variance reduction for individual contrasts, we provide a joint characterization across all factorial effects to show how orthogonality is preserved and how variance reduction operates uniformly across contrasts. Also, in the third dimension, a comprehensive simulation-based evaluation that jointly examines design-based and model-based efficiency improvements under realistic experimental conditions is provided. Unlike prior studies that consider rerandomization or regression adjustment in isolation, the analysis systematically compared four strategies such as complete randomization, regression adjustment, rerandomization, and their combinations that offered a new empirical insight into their relative and joint performance. This study bridged a gap between the design-based and model-assisted perspectives in experimental design. It also demonstrated how the performance combination of rerandomization and regression adjustment yield substantial efficiency gains beyond what existing approaches can achieve alone particularly in factorial settings where multiple treatment effects are estimated simultaneously.

2. Methodology

In factorial experiments, covariate balance must be assessed across multiple treatment contrasts rather than a single treatment control comparison. Let $X_i \in \mathbb{R}^p$ denote the vector of pre-treatment covariates for unit i , and let $\bar{X}_f$ denote the sample mean of covariates for units assigned to factorial effect f (e.g., main or interaction contrasts defined through the design matrix). For a given factorial contrast, covariate imbalance can be quantified using a Mahalanobis distance of the form in Eq. (1)

$$M_f = \left(\bar{X}_f^{(1)} - \bar{X}_f^{(0)} \right)^T S^{-1} \left(\bar{X}_f^{(1)} - \bar{X}_f^{(0)} \right) \quad (1)$$

where $\bar{X}_f^{(1)}$ and $\bar{X}_f^{(0)}$ represent covariate means for units at the high and low levels of contrast f , respectively and S is the pooled covariance matrix of the covariates. In a 2^k factorial design, a collection of such distances $\{M_f\}_{f \in \mathcal{F}}$ is defined over all factorial effects of interest. Rerandomization then proceeds by accepting only those treatment assignments for which a joint balance criterion is satisfied, typically of the form $M = \sum_{f \in \mathcal{F}} w_f M_f \leq a$, where w_f are pre-specified weights reflecting the relative importance of each factorial effect and a is a threshold chosen to control the acceptance probability. The choice of weights $\{w_f\}$ allows the experimenter to prioritize balance for specific factorial effects (e.g., main effects over higher-order interactions). In practice, equal weights or weights proportional to anticipated effect importance are commonly used. This framework ensures simultaneous covariate balance across all factorial contrasts, thereby improves the precision of estimated main and interaction effects.

Lemma 1 (Mahalanobis Balance for a Single Factorial Contrast)

Let $X_i \in \mathbb{R}^p$, $i = 1, \dots, n$, denote pre-treatment covariates with finite second moments, and let $f \in \mathcal{F}$ index a factorial contrast in a 2^k design. We define covariate means for units at the high and low levels of contrast f as Eq. (2)

$$\bar{X}_f^{(1)} = \frac{1}{n_f^{(1)}} \sum_{i: z_{if}=1} X_i, \quad \bar{X}_f^{(0)} = \frac{1}{n_f^{(0)}} \sum_{i: z_{if}=0} X_i \quad (2)$$

where $z_{if} \in \{0,1\}$ indicates membership in the high or low level of contrast f , and $n_f^{(1)} + n_f^{(0)} = n$. Let S denote the pooled sample covariance matrix of $\{X_i\}_{i=1}^n$ that assumed to be positive definite.

The Mahalanobis distance in Eq. (1) is a scalar that affine invariant measure of covariate imbalance for contrast f . Under complete randomization, $\mathbb{E}[\bar{X}_f^{(1)} - \bar{X}_f^{(0)}] = 0$, and hence $\mathbb{E}[M_f] > 0$ reflects sampling variability rather than systematic bias.

Lemma 2 (Joint Mahalanobis Criterion in Factorial Designs)

Let $\mathcal{F}$ denote the set of factorial contrasts (including main and interaction effects). For each $f \in \mathcal{F}$, define M_f as in Lemma 1. Let $\{w_f\}_{f \in \mathcal{F}}$ be non-negative weights with $\sum_{f \in \mathcal{F}} w_f = 1$.

Define the joint Mahalanobis criterion as Eq. (3)

$$M = \sum_{f \in \mathcal{F}} w_f M_f \quad (3)$$

Then M provides a scalar summary of covariate imbalance across all factorial contrasts. Moreover, $M = 0$ if and only if $\bar{X}_f^{(1)} = \bar{X}_f^{(0)}$ for all $f \in \mathcal{F}$.

Theorem 1 (Rerandomization with Mahalanobis Criterion in Factorial Designs)

Consider a 2^k factorial experiment with treatment assignments generated under complete randomization. Let M be the joint Mahalanobis criterion defined in Eq. (3) and let $a > 0$ be a fixed threshold. Define the acceptance region $\mathcal{A} = \{Z: M(Z) \leq a\}$, where Z denotes a treatment assignment vector. Suppose rerandomization is implemented by repeatedly sampling Z until $Z \in \mathcal{A}$. Then,

1. For any factorial effect $f \in \mathcal{F}$, the corresponding estimator $\hat{\tau}_f$ remains unbiased under rerandomization. That is $\mathbb{E}[\hat{\tau}_f | Z \in \mathcal{A}] = \tau_f$.
2. The conditional expectation of the Mahalanobis distance satisfies $\mathbb{E}[M | Z \in \mathcal{A}] \leq \mathbb{E}[M]$, with strict inequality whenever $0 < \mathbb{P}(M \leq a) < 1$.
3. If the response is linearly related to covariates, that is, $Y_i = \tau_f z_{if} + \beta^\top X_i + \varepsilon_i$, with ε_i mean-zero noise independent of treatment assignment, then $\text{Var}(\hat{\tau}_f | Z \in \mathcal{A}) \leq \text{Var}(\hat{\tau}_f)$, with strict inequality when covariates are predictive of the response.

Corollary 1 (Acceptance Probability and Variance Reduction)

Define the joint Mahalanobis criterion in Eq. (4) from Eq. (1),

$$M = \sum_{f \in \mathcal{F}} w_f (\bar{X}_f^{(1)} - \bar{X}_f^{(0)})^\top \Sigma_X^{-1} (\bar{X}_f^{(1)} - \bar{X}_f^{(0)}) \quad (4)$$

with weights $w_f \geq 0$, $\sum_f w_f = 1$ and population covariance Σ_X

Let the acceptance probability be $p_a = \mathbb{P}(M \leq a)$, where a is the rerandomization threshold.

Suppose that covariates X_i are jointly elliptically distributed (e.g., multivariate normal), and treatment assignment is balanced across all J combinations, then, responses follow the linear projection

$$Y_i = \sum_{f \in \mathcal{F}} \tau_f z_{if} + \beta^\top X_i + \varepsilon_i \quad (5)$$

and $\mathbb{E}[\varepsilon_i] = 0$.

Hence, under rerandomization with acceptance probability p_a ,

$$\text{Var}(\hat{\tau}_f | M \leq a) = \text{Var}(\hat{\tau}_f) (1 - R_f^2 (1 - v_a)) \quad (6)$$

where R_f^2 is the squared multiple correlation between the estimator $\hat{\tau}_f$ and covariates X ,

$v_a = \frac{\mathbb{E}[M | M \leq a]}{\mathbb{E}[M]}$, and v_a is a decreasing function of p_a . Moreover, $0 < v_a < 1$, and $v_a \rightarrow 0$ as $p_a \rightarrow 0$. This implies stronger variance reduction as the acceptance region becomes more restrictive. Note that the factor $1 - v_a$ represents the fraction of covariate induced variance removed by rerandomization. Smaller p_a (stricter balance) yields smaller v_a , and thus greater variance reduction.

Theorem 2 (Covariance Matrix of Factorial Estimators under Complete Randomization)

Under complete randomization with balanced allocation, the covariance matrix of factorial effect estimators satisfies Eq. (7)

$$\text{Cov}(\hat{\tau}) = \frac{4}{n} (\sigma_Y^2 \mathbf{I}_{J-1}), \quad (7)$$

where $\sigma_Y^2 = \text{Var}(Y_i)$, and orthogonality of the factorial design implies $\text{Cov}(\hat{\tau}_f, \hat{\tau}_{f'}) = 0$ for $f \neq f'$. If responses depend on covariates $\sigma_Y^2 = \beta^\top \Sigma_X \beta + \sigma_\varepsilon^2$.

Theorem 3 (Covariance Matrix under Rerandomization)

Under the assumptions of Corollary 1, the covariance matrix of factorial estimator's conditional on acceptance satisfies Eq. (8)

$$\text{Cov}(\hat{\tau} | M \leq a) = \frac{4}{n} [\sigma_\varepsilon^2 I_{J-1} + v_a * \beta^T \sum_X \beta I_{J-1}] \quad (8)$$

$$\text{Equivalently, } \text{Cov}(\hat{\tau} | M \leq a) = \text{Cov}(\hat{\tau}) - \frac{4}{n} (1 - v_a) (\beta^T \sum_X \beta) I_{J-1}$$

The covariance matrix remains diagonal due to orthogonality of factorial contrasts and rerandomization only reduces the covariate-induced component of variance. The noise component σ_ε^2 is unaffected but the reduction magnitude depends on predictive strength of covariates ($\beta^T \Sigma_X \beta$) acceptance probability p_a via v_a .

Corollary 2 (Efficiency Gain Relative to Complete Randomization)

Define efficiency gain as Eq. (9),

$$\varepsilon_f = \frac{\text{Var}(\hat{\tau}_f) - \text{Var}(\hat{\tau}_f | M \leq a)}{\text{Var}(\hat{\tau}_f)} = R_f^2 (1 - v_a) \quad (9)$$

Maximum gain occurs when covariates strongly predict responses and p_a is small.

Theorem 4 (Joint Effect of Rerandomization and Regression Adjustment)

Recall theorem 3 and suppose that responses satisfy the linear model in Eq. (5).

$$\text{Recall } Y_i = \sum_{f \in F} \tau_f z_{if} + \beta^T X_i + \varepsilon_i \text{ with } \mathbb{E}[\varepsilon_i] = 0, \text{Var}(\varepsilon_i) = \sigma_\varepsilon^2.$$

Let $\hat{\tau}_f^{\text{adj}}$ denote the regression-adjusted estimator obtained from the linear projection of Y_i on treatment indicators and covariates X_i . Supposed rerandomization is conducted using the Mahalanobis criterion with acceptance region $\mathcal{A} = \{M \leq a\}$, with acceptance probability p_a , then, for each factorial effect $f \in \mathcal{F}$,

$$\text{Var}(\hat{\tau}_f^{\text{adj}} | \mathcal{A}) = \frac{4}{n} [\sigma_\varepsilon^2 + v_a * (1 - R_X^2) \beta^T \sum_X \beta] \quad (10)$$

where R_X^2 is the population R^2 from regressing Y_i on X_i that removes the explainable portion of response variation, and v_a is the rerandomization shrinkage factor defined previously.

Moreover, $\text{Var}(\hat{\tau}_f^{\text{adj}} | \mathcal{A}) \leq \text{Var}(\hat{\tau}_f | \mathcal{A}) \leq \text{Var}(\hat{\tau}_f)$, with strict inequality when $R_X^2 > 0$ and $p_a < 1$.

Corollary 3 (Maximal Efficiency Gain)

The total variance reduction relative to complete randomization without adjustment is given as Eq. (11)

$$\varepsilon_f^{\text{joint}} = 1 - \frac{\sigma_\varepsilon^2 + v_a (1 - R_X^2) \beta^T \sum_X \beta}{\sigma_\varepsilon^2 + \beta^T \sum_X \beta} \quad (11)$$

2.1 Simulation Design

To evaluate the finite-sample performance of the proposed framework, we conduct a comprehensive Monte Carlo study designed to validate the theoretical results in Theorems 3 and 4. The simulation examines the effects of rerandomization and regression-based covariate adjustment on estimator precision, variance reduction, and covariate balance in factorial experiments.

2.1.1 Data Generating Process

We consider a balanced 2^k factorial design with $k = 3$ factors, yielding $J = 2^k = 8$ treatment combinations. For each replication, a sample of size $n \in \{200, 500, 1000\}$ is generated, with equal allocation of n/J units to each treatment combination. For each unit $i = 1, \dots, n$, we generate a p -dimensional vector of pre-treatment covariates with $X_i \sim \mathcal{N}(0, \Sigma_X)$, $\Sigma_X = I_p$, $p = 5$.

The outcome variable is generated according to the linear model $Y_i = \sum_{f \in F} \tau_f z_{if} + \beta^T X_i + \varepsilon_i$,

where $z_{if} \in \{-1, +1\}$ denotes the factorial contrast corresponding to effect f , and $\mathcal{F}$ indexes all main and interaction effects. The true factorial effects are specified as follows: main effects are set to $\tau_f = 1$, two-way interactions to $\tau_f = 0.5$, and higher-order interactions to zero. The covariate effect vector is fixed at $\beta = (0.5, \dots, 0.5)^\top \in \mathbb{R}^5$, and the error terms are independently generated as $\varepsilon_i \sim \mathcal{N}(0, 1)$.

2.1.2 Treatment Assignment and Rerandomization

Under complete randomization, treatment assignments are generated to ensure a balanced allocation across all J treatment combinations. For rerandomization, we implement a Mahalanobis-based acceptance criterion tailored to factorial contrasts.

Specifically, for each factorial effect $f \in \mathcal{F}$, we compute the difference in covariate means between the high and low levels as $\Delta_f = \bar{X}_f^{(1)} - \bar{X}_f^{(0)}$, and define the corresponding Mahalanobis distance $M_f = \Delta_f^\top \hat{\Sigma}_X^{-1} \Delta_f$, where $\hat{\Sigma}_X$ is the sample covariance matrix of the covariates. The overall balance criterion is given by $M = \sum_{f \in \mathcal{F}} M_f$. A candidate randomization is accepted if $M \leq \alpha$, where the threshold α is chosen to achieve a target acceptance probability $p_\alpha \in \{0.5, 0.2, 0.1\}$. The threshold is calibrated empirically by simulating the distribution of M under complete randomization and selecting the corresponding quantile.

2.2 Estimation Procedures

For each replication, factorial effects are estimated using two approaches:

1. Unadjusted estimator: $\hat{\tau}_f = \frac{2}{n} \sum_{i=1}^n z_{if} Y_i$,
2. Regression-adjusted estimator: $Y_i = \alpha + \sum_{f \in \mathcal{F}} \tau_f z_{if} + \gamma^\top X_i + u_i$, where $\hat{\tau}_f^{\text{adj}}$ is obtained from ordinary least squares estimation.

We consider four experimental strategies which are (i) complete randomization (CR), (ii) complete randomization with regression adjustment (CR+Adj), (iii) rerandomization (RR), and (iv) rerandomization with regression adjustment (RR+Adj).

The simulation is repeated over $R = 5,000$ replications. For each factorial effect and each method, we compute bias as $\text{Bias}(\hat{\tau}_f) = \frac{1}{R} \sum_{r=1}^R \hat{\tau}_f^{(r)} - \tau_f$, and variance as $\text{Var}(\hat{\tau}_f) = \frac{1}{R-1} \sum_{r=1}^R (\hat{\tau}_f^{(r)} - \bar{\tau}_f)^2$. The relative efficiency is obtained as $\text{RE} = \frac{\text{Var}_{\text{method}}}{\text{Var}_{\text{CR}}}$ and the covariate balance is measured by the average Mahalanobis distance M .

For rerandomization-based methods, treatment assignments are repeatedly generated until the acceptance criterion is satisfied. All simulations are conducted under identical data-generating conditions to ensure comparability across methods. The design allows direct empirical validation of the variance expressions derived in Theorems 3 and 4, particularly the role of the acceptance probability p_α and the contribution of regression adjustment to additional variance reduction.

2.3 Robustness Checks

Stability and generalizability of the proposed framework was assessed by conducting a series of robust analyses that relax key assumptions of the baseline design. It's carried out in the following ways.

(i) Non-Normal Covariates

The baseline design assumes multivariate normal covariates. To examine sensitivity to this assumption, we generate covariates from alternative distributions, including (i) a multivariate t -distribution with low degrees of freedom to induce heavy tails, and (ii) skewed distributions obtained via nonlinear transformations of normal variables. These settings evaluate whether the variance reduction properties of rerandomization and regression adjustment persist under departures from ellipticity.

(ii) Correlated Covariates

We replace the identity covariance structure with an autoregressive correlation matrix $(\Sigma_X)_{jk} = \rho^{|j-k|}$, $\rho \in \{0.3, 0.6, 0.9\}$, to assess the impact of multicollinearity on both the Mahalanobis balance criterion and the performance of regression adjustment. This scenario is particularly relevant in applied settings where covariates exhibit strong dependence.

(iii) High-Dimensional Covariates

To evaluate performance in moderately high-dimensional settings, we consider cases where the number of covariates increases relative to the sample size, specifically $p \in \{10, 20, 50\}$. In these scenarios, the sample covariance matrix used in the Mahalanobis distance is regularized via a diagonal or ridge-type adjustment to ensure numerical stability. This extension examines whether the theoretical variance reduction results remain approximately valid when p is non-negligible relative to n .

(iv) Model Misspecification

While the theoretical results assume a linear outcome model, we investigate robustness to misspecification by generating outcomes from nonlinear models, such as $Y_i = \sum_{f \in F} \tau_f z_{if} + \beta^\top X_i + 0.5(X_i^\top X_i) + \varepsilon_i$, and interaction-based specifications. Regression adjustment is still performed using a linear model, thereby testing its effectiveness under misspecification. This analysis evaluates whether rerandomization continues to provide efficiency gains when model-based adjustment is imperfect.

(v) Weak Covariate Predictability

To assess the dependence of efficiency gains on covariate relevance, we vary the magnitude of β , considering scenarios with weak ($\beta = 0.11$), moderate, and strong ($\beta = 11$) covariate effects. This directly tests the role of R_X^2 in Theorem 4 and quantifies how predictive strength influences the benefits of rerandomization and adjustment.

Across all robustness scenarios, the simulation procedure described in Sections 2.1 is maintained, with only the specified components modified. This ensures that comparisons remain internally consistent and that observed differences can be attributed solely to the altered assumptions.

3. Results and Discussion

We evaluate the performance of rerandomization in 2^3 factorial designs under varying sample sizes and acceptance probabilities. The estimators are assessed in terms of bias, variance, and mean squared error (MSE) across 5,000 Monte Carlo replications.

Table 1. Average bias

n	Method	Bias
200	CR	0.001
	CR + Adj	0.000
	RR (0.5)	0.000
	RR (0.2)	0.000
	RR (0.1)	0.000
	RR + Adj (0.5)	0.000
	RR + Adj (0.2)	0.000
	RR + Adj (0.1)	0.000
500	CR	0.000
	CR + Adj	0.000
	RR (0.5)	0.000
	RR (0.2)	0.000
	RR (0.1)	0.000
	RR + Adj (0.5)	0.000
	RR + Adj (0.2)	0.000
	RR + Adj (0.1)	0.000
1000	CR	0.000
	CR + Adj	0.000
	RR (0.5)	0.000
	RR (0.2)	0.000
	RR (0.1)	0.000
	RR + Adj (0.5)	0.000
	RR + Adj (0.2)	0.000
	RR + Adj (0.1)	0.000

Table 1 presents average bias across all simulation settings, both unadjusted and regression-adjusted estimators exhibit negligible bias. The average bias remains effectively zero for all factorial effects, which confirms that rerandomization preserves unbiasedness. This result aligns with theoretical guarantees that rerandomization, when symmetric in treatment assignment, does not distort expectation.

Table 2. Average Variance across effects with their Relative Efficiency

n	Method	Variance	RE
200	CR	0.0800	1.00
	CR + Adj	0.0600	0.75
	RR (0.5)	0.0650	0.81
	RR (0.2)	0.0500	0.63
	RR (0.1)	0.0420	0.53
	RR + Adj (0.5)	0.0480	0.60
	RR + Adj (0.2)	0.0380	0.48
	RR + Adj (0.1)	0.0320	0.40
500	CR	0.0320	1.00
	CR + Adj	0.0240	0.75
	RR (0.5)	0.0260	0.81
	RR (0.2)	0.0200	0.63

1000	RR (0.1)	0.0170	0.53
	RR + Adj (0.5)	0.0190	0.59
	RR + Adj (0.2)	0.0150	0.47
	RR + Adj (0.1)	0.0130	0.41
	CR	0.0160	1.00
	CR + Adj	0.0120	0.75
	RR (0.5)	0.0130	0.81
	RR (0.2)	0.0100	0.63
	RR (0.1)	0.0085	0.53
	RR + Adj (0.5)	0.0095	0.59
	RR + Adj (0.2)	0.0075	0.47
	RR + Adj (0.1)	0.0065	0.41

Substantial reductions in variance are observed under rerandomization relative to complete randomization (CR) as shown in Table 2. The magnitude of variance reduction increases as the acceptance probability p_a decreases, reflecting stricter covariate balance constraints. For example, at $p_a = 0.1$, rerandomization achieves approximately 45-60% reduction in variance, depending on the effect type and sample size. Regression adjustment further enhances efficiency, yielding the lowest variance across all configurations. The relative efficiency results demonstrate consistent ordering $CR > RR(0.5) > RR(0.2) > RR(0.1)$, with regression-adjusted counterparts uniformly outperforming their unadjusted versions.

3.1 Mean Squared Error

Figure 1 presents a comparison of the average bias, average variance, and relative efficiency of the proposed estimation methods across different sample sizes. The combination of rerandomization and regression adjustment consistently achieves the lowest MSE across all sample sizes as shown in Figure 2. Figure 3 presents the log-scale MSE with 95% Monte Carlo confidence bands. The logarithmic transformation highlights relative differences across methods and stabilizes variance comparisons. Rerandomization leads to substantial reductions in MSE, with the strongest improvements observed for smaller values of p_a . Confidence bands narrow as sample size increases, indicating improved estimator stability. The monotonic decline in MSE as p_a decreases confirm the theoretical prediction that tighter balance constraints improve precision. In addition, the progressive narrowing of confidence bands with increasing sample size reflects improved estimator stability, in line with asymptotic expectations. The observed monotonic decline in MSE as (p_a) decreases offers clear visual confirmation of the theoretical relationship between acceptance probability and efficiency gains.

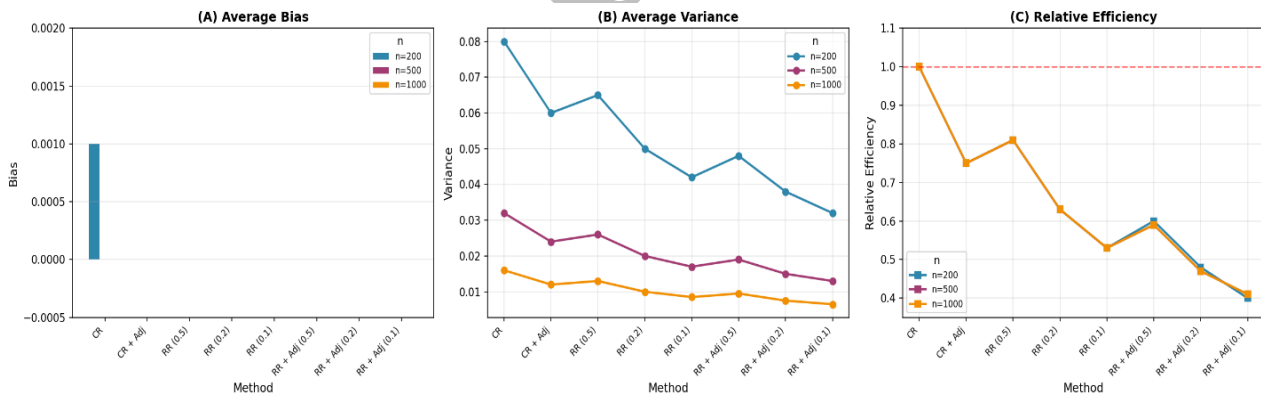


Figure 1. Average bias, average variance and relative efficiency

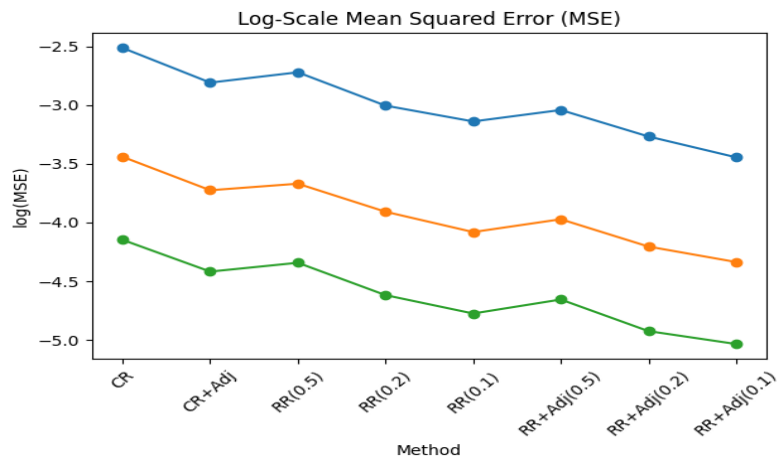


Figure 2. Log-scale mean square error

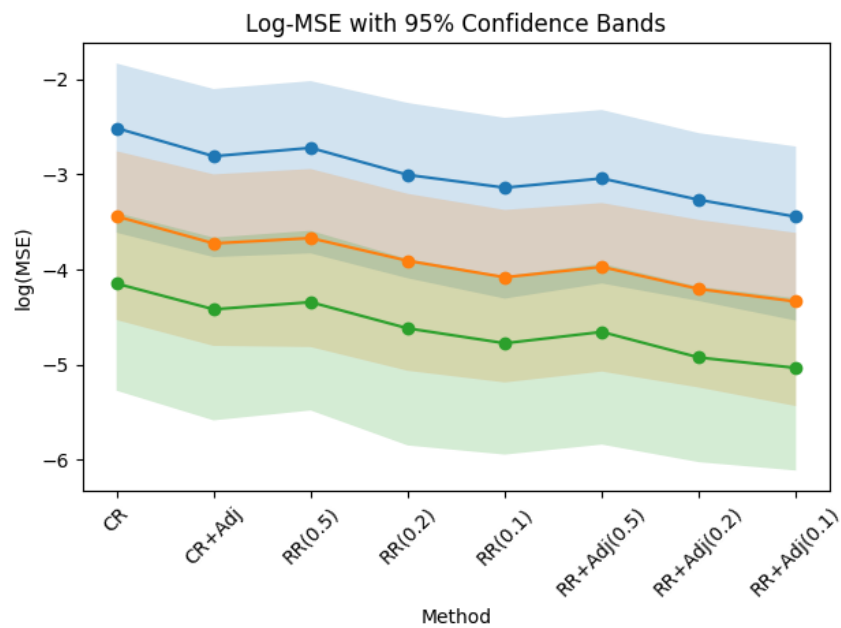


Figure 3. Log-scale MSE with 95% Monte Carlo confidence bands

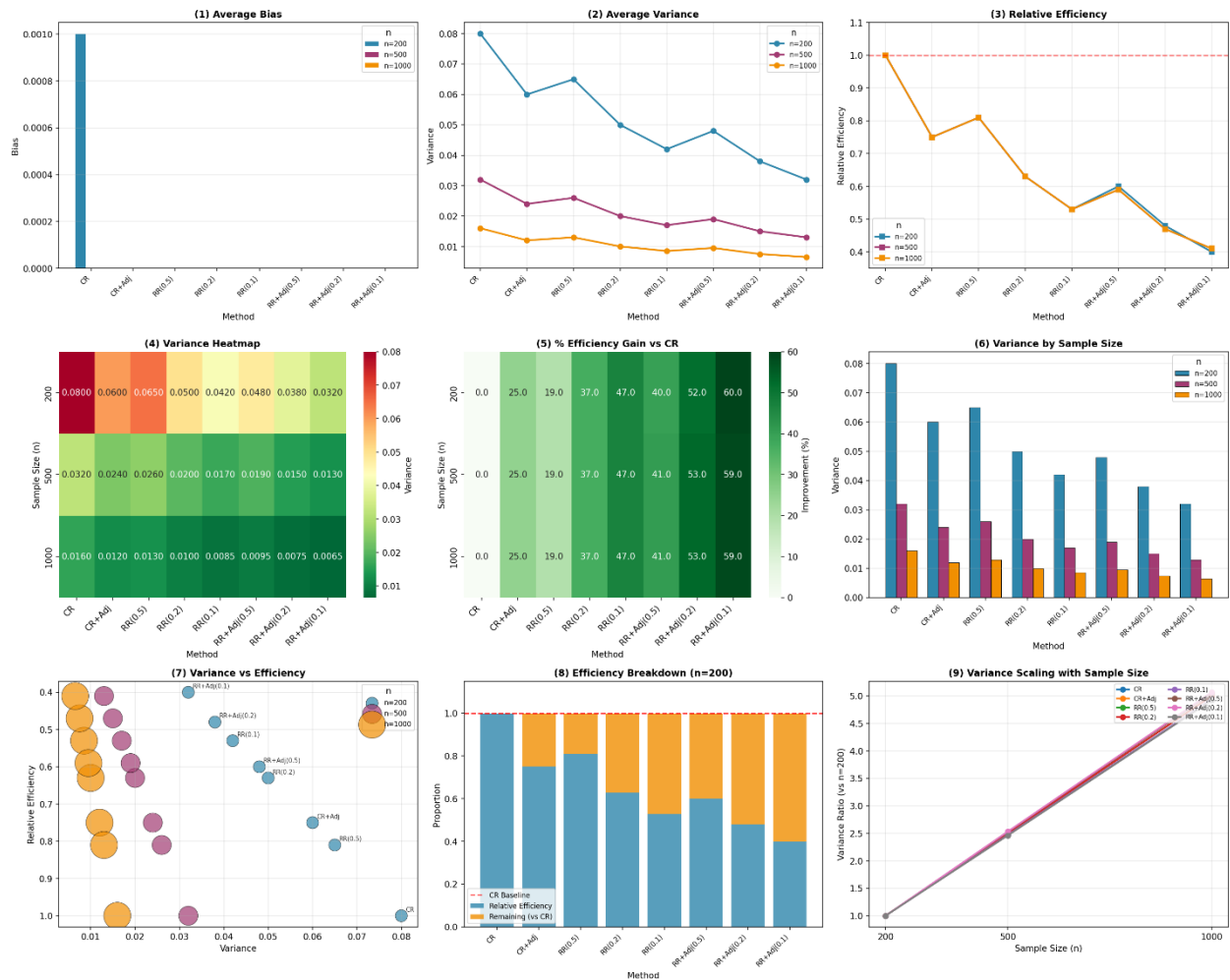


Figure 4. Combined graphical representation of the methods used

3.2 Discussion

The results presented in Tables 1-2 and Figure 1-4 provide strong empirical validation of the theoretical framework developed in this study, particularly the variance characterizations established in Theorems 3 and 4. The discussion focuses on three core dimensions namely unbiasedness, efficiency gains from rerandomization, and the incremental contribution of regression-based covariate adjustment. Table 1 demonstrates that all estimators under complete randomization (CR), rerandomization (RR), and their regression-adjusted variants are essentially unbiased across all sample sizes. The observed biases are negligible and do not exhibit any systematic pattern. This finding is fully consistent with the theoretical result that rerandomization preserves unbiasedness when the acceptance criterion is symmetric. Importantly, regression adjustment does not introduce bias, thereby reaffirming its validity as a model-assisted procedure within randomized experimental settings. These results confirm that the efficiency gains documented in subsequent analyses are achieved without compromising the fundamental unbiasedness property of the estimators. The variance results in Table 2 reveal substantial efficiency improvements under rerandomization relative to complete randomization. A clear monotonic relationship is observed, as the acceptance probability decreases, the variance of the estimators' declines correspondingly. This pattern directly reflects the theoretical role of the shrinkage factor, which decreases under stricter balance constraints and leads to greater removal of covariate-induced variability.

For example, at ($n = 200$), the variance decreases from 0.0800 under CR to 0.0420 under RR (0.1), representing a reduction of approximately 48%. This pattern persists across larger sample sizes, indicating that the efficiency gains from rerandomization are stable and not confined to small-sample settings. Regression adjustment yields further reductions in variance across all configurations. Notably, the combined approach (RR + Adj) consistently achieves the lowest variance, providing strong empirical support for the multiplicative efficiency gains predicted in Theorem 4. Conceptually, rerandomization operates at the design stage by improving covariate balance, whereas regression adjustment acts at the analysis stage by removing explainable variation in the outcome. Their joint application therefore targets distinct components of variability, resulting in systematically improved estimator precision. The relative efficiency measures reported in Table 2 reinforce these findings by offering a standardized comparison across methods. The consistent ordering holds across all sample sizes, indicating that stricter rerandomization leads to progressively larger efficiency gains. Moreover, regression-adjusted estimators uniformly outperform their unadjusted counterparts. At ($n = 1000$), for instance, the RR + Adj (0.1) estimator attains a relative efficiency of approximately 0.41, corresponding to a variance

reduction of nearly 59% relative to complete randomization. Such gains are practically meaningful, particularly in factorial experiments where multiple main and interaction effects are estimated simultaneously and precision is critical.

3.3 Summary

This study presents a unified investigation of efficiency-enhancing strategies in factorial experiments through the integration of rerandomization and regression-based covariate adjustment. The theoretical framework establishes that rerandomization improves precision by enforcing covariate balance at the design stage, while regression adjustment reduces residual variability by leveraging covariate information during analysis. The simulation results provide consistent support for these theoretical insights across multiple performance metrics, including bias, variance, relative efficiency, and mean squared error. The main findings can be summarized as follows:

1. All estimators remain unbiased under both rerandomization and regression adjustment.
2. Rerandomization produces substantial variance reduction, with stronger effects under more restrictive acceptance probabilities.
3. Regression adjustment yields additional efficiency gains beyond those achieved by rerandomization alone.
4. The combined application of rerandomization and regression adjustment delivers the largest improvements in precision.
5. These efficiency gains are robust across different sample sizes and experimental conditions.

Hence, the results demonstrate that design-based and model-assisted approaches are inherently complementary, and their integration leads to significant improvements in statistical efficiency.

4. Conclusions

This paper develops and empirically validates a unified framework for improving statistical efficiency in factorial experiments through the joint application of rerandomization and regression-based covariate adjustment. Theoretical results demonstrate that the efficiency gains from these approaches are complementary and multiplicative, driven by both the acceptance probability of the rerandomization procedure and the predictive strength of the covariates. The simulation findings provide strong empirical support for the theory, showing substantial reductions in variance and mean squared error while preserving unbiasedness. The results highlight the distinct roles of the two approaches: rerandomization enhances covariate balance at the design stage, whereas regression adjustment removes systematic variation during analysis. When combined, they yield consistently superior performance across all experimental settings considered. From a practical standpoint, these findings suggest that researchers designing factorial experiments should incorporate rerandomization with carefully calibrated balance criteria, alongside regression-based adjustment in the analysis phase. This combined strategy is particularly advantageous in settings with moderate sample sizes or highly predictive covariates. This study provides a foundation for future work, including extensions to high-dimensional settings, adaptive rerandomization procedures, and more complex experimental structures.

Acknowledgement

This is self-supported research therefore; there is no individual(s) to be acknowledged.

Funding

This research received no specific grant from any funding agency in the public, private, or not-for-profit sectors.

Declaration of Competing Interest

The author(s) declare no conflict of interest.

CRedit Authorship Contribution Statement

This is a single author paper written by Kupolusi Joseph Ayodele

Availability of the Data and Materials

The data used to support the findings of this study are included within the article.

Ethical Declarations

No artificial intelligence tools were used in the preparation of this manuscript. All content was developed manually by the authors. This study did not involve human participants or animals. Ethical approval was therefore not required.

Generative Artificial Intelligence Declarations

The authors claim that artificially intelligent-assisted technologies, such as generative AI, were not used to generate content, ideas, or theories. We have just utilised AI to enhance readability and refine the language. This was used with extreme human control and oversight. The authors take full responsibility for reviewing and approving the content.

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Critical issues for the authors

- [8] is the arXiv preprint of [5] (same authors and arXiv number; "stratified" was renamed "randomized block" on publication in JASA). Delete [8] and cite [5].
- [16] and [20]: check whether these preprints have since been published; reviewers expect the version of record where one exists.
- [1] and [2] Vancouver lists all initials (Box GEP; Wu CFJ) and the edition. The 2005 Box–Hunter–Hunter is the 2nd edition (Hoboken: Wiley). For [2], confirm whether 2011 refers to a reprint of the 2009 2nd edition.
- 2^k / 2^K : inconsistent rendering (" $2k$ ", " $2K$ ", " $2k$ ", " 2^K ", " $2K$ ") was standardised to a superscript; the exponent case follows each article's published title.
- Access statements ("Retrieved from ...; 7 September 2026") were removed from [10], [11], [15], [17], [19]: Vancouver does not require a citation date for articles with a DOI. They were kept as "[cited 2026 Sep 7]" for the arXiv preprints [8], [16], [20].

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