

RESEARCH ARTICLE

An adaptive step-size modified superclass block method for stiff and oscillatory differential systems

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Abstract - This paper presents a modified adaptive step-size superclass of block backward differentiation formula (MVSBBDF) for the numerical integration of stiff and oscillatory differential systems. The proposed method extends the two-point variable step-size superclass of block BDF by introducing a positive corrective term that enhances both accuracy and stability. The MVSBBDF computes two solution values simultaneously at each integration step while dynamically adjusting the step size based on local truncation error control. A predictor formula is incorporated to provide starting values for the corrector phase, ensuring robust and efficient convergence. Theoretical analyses establish that the method is of order four, zero-stable, A-stable, and convergent. The step-size selection strategy allows the algorithm to adapt efficiently to stiffness variations within the solution domain. The effectiveness of the proposed method is demonstrated through several stiff and oscillatory test problems. Numerical results confirm that the MVSBBDF delivers improved accuracy, stable convergence, and minimal step rejections while maintaining competitive computational efficiency. Overall, the method provides an effective and robust framework for the numerical solution of stiff systems of ordinary differential equations with adaptive step-size control.

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1. Introduction

Stiff and oscillatory differential systems are pervasive in various fields of science and engineering, such as chemical reaction kinetics with widely varying reaction rates, electrical circuits and power systems, mechanical vibrations with damping and external forces, as well as wave propagation phenomena [1-2]. These equations pose significant numerical challenges due to the coexistence of rapid temporal oscillations and multiple scales of decay or growth rates, which characterize stiffness. Consequently, achieving efficient and accurate numerical integration of such systems demands methods that can effectively balance stability, precision (particularly in preserving oscillatory behaviour), and computational efficiency [3]. Stiffness is generally described by the wide variation in the eigenvalues of the system's Jacobian matrix, where the presence of eigenvalues with large negative real parts compels explicit numerical methods to adopt very small step sizes to maintain stability [4-5]. In contrast, implicit methods possess better stability characteristics, allowing them to use larger step sizes, which make them more suitable for stiff systems [6]. The foundational understanding of stiffness and the justification for using implicit solvers can be traced to the pioneering work of Curtiss and Hirschfelder [7], which later inspired the development of stiffly stable numerical techniques such as the Backward Differentiation Formulas (BDF), Extended Backward Differentiation Formulas (EBDF), Superclass of Extended Backward Differentiation Formulas (SEBDF) and their modified versions [8-10]. These approaches continue to form the core of many modern stiff problem solvers. Oscillatory behaviour presents a unique challenge because numerical methods must accurately maintain both the phase and amplitude of oscillations over extended time intervals. If this is not achieved, the computed solution may experience phase errors and amplitude attenuation, leading to a significant loss of accuracy [11]. For problems that are purely oscillatory and not stiff, specialized numerical approaches such as trigonometrically fitted integrators, and exponential integrators, have been developed to minimize phase errors while allowing relatively large time steps. When oscillations occur together with stiff decay modes, however, the numerical method must simultaneously ensure stability for stiffness and precision for oscillatory behaviour, making the construction of efficient algorithms considerably more difficult [3].

A practical strategy for addressing problems that exhibit both stiffness and oscillatory behaviour involves the use of implicit or semi-implicit integration techniques that maintain A-stability or possess wide stability regions. These approaches are often enhanced with adaptive mechanisms or corrections designed to better handle oscillatory dynamics. Implicit multistep and block-based methods, such as block backward differentiation formulas (BBDF) and their superclass extensions, provide the advantage of computing multiple solution points within a single integration step

while reusing Jacobian and linear system computations [12-16]. This feature improves both stability and computational efficiency when solving stiff or oscillatory systems. The superclass formulation, in particular, introduces additional parameters that define a family of related numerical schemes, enabling researchers to adjust coefficients to achieve a suitable balance between stability and accuracy. Suleiman et al. [17] and Musa et al. [18] developed the class of fully implicit superclass of block backward differentiation methods (SBBDF) and showed that they offer greater accuracy for stiff initial value problems, although they require more function evaluations per step. However, these studies developed diagonally implicit and off-step extensions to reduce computational costs while maintaining the robust stability characteristics of the superclass framework [19-22]. A general first order stiff oscillatory initial value problem (IVP) can be expressed as [3]:

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \leq x \leq b. \quad (1)$$

where the function f satisfies the Lipschitz continuity condition with respect to y . The existence and uniqueness of the solution to equation (1) are ensured by the Lipschitz condition. Specifically, there exists a constant $L > 0$ such that

$$\|f(x, y_1) - f(x, y_2)\| \leq \|y_1 - y_2\|, \quad \forall x \in [a, b], y_1, y_2 \in \mathbb{R}^n.$$

This condition guarantees that the function $f(x, y)$ does not vary too rapidly with respect to y , thereby ensuring the well-posedness of the IVP and the uniqueness of its solution.

Adaptivity, often implemented through variable step-size control, plays a vital role in the efficient numerical integration of stiff and oscillatory differential equations. By adjusting the step size dynamically, the integrator can take larger steps in regions where the solution changes gradually and smaller steps in regions characterized by rapid oscillations or transient behaviour. This flexibility enhances computational efficiency while maintaining the desired level of local accuracy [23]. However, applying variable step sizes in multistep and block methods introduces additional complexity, since non-uniform time grids alter the recurrence coefficients and may affect both 0-stability and the overall order and error constant of the scheme. To overcome these challenges, researchers have developed robust step-size ratio controls and coefficient update procedures to preserve convergence and stability. Different studies found in the literature have contributed significantly to this area by proposing variable step-size block backward differentiation formulas and adaptive algorithms that enforce safe step-ratio limits and employ embedded error estimation for step-size adjustment [24-29]. These approaches have demonstrated improved accuracy and efficiency in solving stiff and oscillatory IVPs. From a theoretical standpoint, ensuring zero-stability, consistency, and A-stability for variable-step block and superclass methods requires:

- i) Satisfying the usual order conditions via Taylor expansions for the desired order,
- ii) Controlling step-ratio variation so that the root condition for the associated characteristic polynomials remains satisfied on non-uniform grids, and
- iii) Analysing the method's stability polynomial against the standard first order linear test equation $y' = \lambda y$.

The objective of this paper is to construct a modified fully implicit, stiffly stable, adaptive step-size superclass of block numerical scheme for stiff and oscillatory differential systems. This development is based on the variable step-size two-point superclass of BBDF introduced by [26], expressed as:

$$\sum_{j=0}^4 \alpha_{j,i,r} y_{n+j-2} = h \beta_{k,i,r} (f_{n+k} - \rho f_{n+k-1}), \quad k = 1, 2. \quad (2)$$

where y_{n+j-2} are the previous and current approximations, $f_{n+k} = f(x_{n+k}, y_{n+k})$, h denotes the block step size, and $\alpha_{j,i,r}$, $\beta_{k,i,r}$ are coefficients that depend on the block index i and step-size ratio r . To improve the stability, accuracy, and computational performance of equation (2), modification is proposed by replacing the term $-\rho f_{n+k-1}$ with $+\rho f_{n+k-1}$. The resulting modified scheme is expressed as

$$\sum_{j=0}^4 \alpha_{j,i,r} y_{n+j-2} = h \beta_{k,i,r} (f_{n+k} + \rho f_{n+k-1}), \quad k = i = 1, 2. \quad (3)$$

This formulation, referred to as the adaptive step-size modified superclass block backward differentiation method (MVSBBDF), retains the block property of computing two solution values per step while introducing a structural adjustment that aims to expand the stability region and enhance numerical efficiency. It is noteworthy that the proposed method simplifies to the conventional variable step-size Block BDF (VSBDF) introduced by Ibrahim et al. [24, 25] when the parameter ρ is set to zero, provided that ρ lies in the stiff-stability range $\rho \in (-1, 1)$.

2. Methodology

Given three previous values at y_{n-2} , y_{n-1} and y_n , the objective is to derive a block formula capable of simultaneously generating the next two approximations y_{n+1} and y_{n+2} . This formulation defines the next block of solutions within the framework of the variable step-size superclass approach. To maintain zero stability while achieving high accuracy and computational efficiency, the variation of the step size between successive blocks is carefully controlled. The step size associated with the current block is denoted by $2h$, while that of the preceding block is $2rh$, where r represents the step-size ratio between two consecutive blocks, as illustrated in Figure 1. In this study, the parameter r is taken to be $r = 1$, $r = 2$ and $r = \frac{5}{8}$, corresponding, respectively, to a constant step size, halving of the step size, and an increase of the step size by a factor of 1.6.

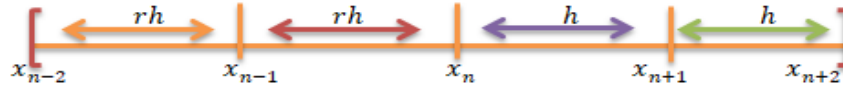


Figure 1. Interpolation points involved in the MVSBBDF method.

2.1 Derivation of the Method

Equation (3) is derived by applying a Taylor series expansion about x_n as follows:

Definition 1: The Linear operator L_i associated with first and second points of the MVSBBDF method is defined as:

$$L_i[y(x_n), h]: \alpha_{0,i}y(x_n - 2rh) + \alpha_{1,i}y(x_n - rh) + \alpha_{2,i}y(x_n) + \alpha_{3,i}y(x_n + h) + \alpha_{4,i}y(x_n + 2h) - h\beta_{k,i}f(x_n + kh) - h\rho\beta_{k,i}f(x_n + (k-1)h) = 0, \quad k = i = 1, 2. \quad (4)$$

To derive the first point, the linear operator corresponding to the first point in (4) becomes

$$L_1[y(x_n), h]: \alpha_{0,1}y(x_n - 2rh) + \alpha_{1,1}y(x_n - rh) + \alpha_{2,1}y(x_n) + \alpha_{3,1}y(x_n + h) + \alpha_{4,1}y(x_n + 2h) - h\beta_{1,1}y'(x_n + h) - h\rho\beta_{1,1}y'(x_n) = 0, \quad (5)$$

Expanding the functions in equation (5) using a Taylor series and collecting like terms yields:

$$C_{0,1}y(x_n) + C_{1,1}hy'(x_n) + C_{2,1}h^2y''(x_n) + C_{3,1}h^3y'''(x_n) + \dots = 0. \quad (6)$$

where,

$$\left. \begin{aligned} C_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} + \alpha_{4,1} = 0 \\ C_{1,1} &= -2r\alpha_{0,1} - r\alpha_{1,1} + \alpha_{3,1} + 2\alpha_{4,1} - \beta_{1,1}(1 + \rho) = 0 \\ C_{2,1} &= 2r^2\alpha_{0,1} + \frac{1}{2}r^2\alpha_{1,1} + \frac{1}{2}\alpha_{3,1} + 2\alpha_{4,1} - \beta_{1,1} = 0 \\ C_{3,1} &= -\frac{4}{3}r^3\alpha_{0,1} - \frac{1}{6}r^3\alpha_{1,1} + \frac{1}{6}\alpha_{3,1} + \frac{4}{3}\alpha_{4,1} - \frac{1}{2}\beta_{1,1} = 0 \\ C_{4,1} &= \frac{2}{3}r^4\alpha_{0,1} + \frac{1}{24}r^4\alpha_{1,1} + \frac{1}{24}\alpha_{3,1} + \frac{2}{3}\alpha_{4,1} - \frac{1}{6}\beta_{1,1} = 0 \end{aligned} \right\}. \quad (7)$$

In deriving the first point y_{n+1} , the coefficient $\alpha_{3,1}$ is normalized to 1. By solving equation (7) in the Maple 18 software environment, the values of $\alpha_{j,i}$ and $\beta_{j,i}$ are obtained as follows:

$$\begin{aligned} \alpha_{0,1} &= \frac{1}{4} \frac{2r\rho - r - 1}{r^2(4r^2\rho + 3r + 2)}, \quad \alpha_{1,1} = -\frac{8r^2\rho - 4r^2 + 4r\rho - 4r - 1}{r^2(4r^3\rho + 8r^2\rho + 3r^2 + 8r + 4)}, \\ \alpha_{2,1} &= -\frac{1}{4} \frac{12r^4\rho + 4r^4 + 6r^3\rho + 12r^3 - 12r^2\rho + 13r^2 - 6r\rho + 6r + 1}{r^2(4r^2\rho + 3r + 2)}, \\ \alpha_{4,1} &= -\frac{1}{4} \frac{4r^3\rho - 4r^3 + 2r^2\rho - 8r^2 - 5r - 1}{(4r^3\rho + 8r^2\rho + 3r^2 + 8r + 4)}, \quad \alpha_{3,1} = 1, \quad \beta_{1,1} = \frac{2r^2 + 3r + 1}{(4r^2\rho + 3r + 2)} \end{aligned}$$

Substituting these values into equation (5) yields the first point as

$$\begin{aligned} y_{n+1} &= -\frac{1}{4} \frac{2r\rho - r - 1}{r^2(4r^2\rho + 3r + 2)} y_{n-2} + \frac{8r^2\rho - 4r^2 + 4r\rho - 4r - 1}{r^2(4r^3\rho + 8r^2\rho + 3r^2 + 8r + 4)} y_{n-1} \\ &+ \frac{1}{4} \frac{12r^4\rho + 4r^4 + 6r^3\rho + 12r^3 - 12r^2\rho + 13r^2 - 6r\rho + 6r + 1}{r^2(4r^2\rho + 3r + 2)} y_n + \frac{1}{4} \frac{4r^3\rho - 4r^3 + 2r^2\rho - 8r^2 - 5r - 1}{(4r^3\rho + 8r^2\rho + 3r^2 + 8r + 4)} y_{n+2} \\ &+ \frac{2r^2 + 3r + 1}{(4r^2\rho + 3r + 2)} hf_{n+1} + \frac{2r^2 + 3r + 1}{(4r^2\rho + 3r + 2)} hf_n \end{aligned} \quad (8)$$

Similarly, to derive the second point y_{n+2} , the linear operator corresponding to this point is defined as

$$L_2[y(x_n), h] = \alpha_{0,2}y(x_n - 2rh) + \alpha_{1,2}y(x_n - rh) + \alpha_{2,2}y(x_n) + \alpha_{3,2}y(x_n + h) + \alpha_{4,2}y(x_n + 2h) - h\beta_{2,2}(y'(x_n + 2h) - \rho y'(x_n + h)) = 0. \quad (9)$$

Expanding equation (9) as a Taylor series about x_n and collecting like terms yields:

$$C_{0,2}y(x_n) + C_{1,2}hy'(x_n) + C_{2,2}h^2y''(x_n) + C_{3,2}h^3y'''(x_n) + \dots = 0, \quad (10)$$

where,

$$\left. \begin{aligned} C_{0,2} &= \alpha_{0,2} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{3,2} + \alpha_{4,2} = 0 \\ C_{1,2} &= -2r\alpha_{0,2} - r\alpha_{1,2} + \alpha_{3,2} + 2\alpha_{4,2} - \beta_{2,2}(1 + \rho) = 0 \\ C_{2,2} &= 2r^2\alpha_{0,2} + \frac{1}{2}r^2\alpha_{1,2} + \frac{1}{2}\alpha_{3,2} + 2\alpha_{4,2} - \beta_{2,2}(2 + \rho) = 0 \\ C_{3,2} &= -\frac{4}{3}r^3\alpha_{0,2} - \frac{1}{6}r^3\alpha_{1,2} + \frac{1}{6}\alpha_{3,2} + \frac{4}{3}\alpha_{4,2} - \beta_{2,2}(2 + \frac{1}{2}\rho) = 0 \\ C_{4,2} &= \frac{2}{3}r^4\alpha_{0,2} + \frac{1}{24}r^4\alpha_{1,2} + \frac{1}{24}\alpha_{3,2} + \frac{2}{3}\alpha_{4,2} - \beta_{2,2}(\frac{4}{3} + \frac{1}{6}\rho) = 0 \end{aligned} \right\} \quad (11)$$

In deriving the second point y_{n+2} , the coefficient $\alpha_{4,2}$ is normalized to 1. By solving the simultaneous system in equation (11), the values of $\alpha_{j,i}$ and $\beta_{j,i}$ are obtained as follows:

$$\alpha_{0,2} = -\frac{r^2\rho - 2r^2 + 3r\rho - 8r + 2\rho - 8}{r^2(2r+1)(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}, \alpha_{1,2} = \frac{4(2r\rho - 4r + \rho - 4)}{r^2(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}, \alpha_{2,2} = -\frac{2r^4\rho - 2r^4 + 9r^3\rho - 12r^3 + 14r^2\rho - 26r^2 + 9r\rho - 24r + 2\rho - 8}{r^2(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}, \alpha_{3,2} = 1, \alpha_{4,2} = -\frac{4(4r^3\rho - 3r^2\rho + 20r^2 - 8r\rho + 32r - 4\rho + 16)}{(2r+1)(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}, \beta_{2,2} = \frac{4(r^2 + 3r + 2)}{(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}$$

Substituting these values into equation (9) yields:

$$\begin{aligned} y_{n+2} &= \frac{r^2\rho - 2r^2 + 3r\rho - 8r + 2\rho - 8}{r^2(2r+1)(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}y_{n-2} - \frac{4(2r\rho - 4r + \rho - 4)}{r^2(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}y_{n-1} \\ &+ \frac{2r^4\rho - 2r^4 + 9r^3\rho - 12r^3 + 14r^2\rho - 26r^2 + 9r\rho - 24r + 2\rho - 8}{r^2(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}y_n + \frac{4(4r^3\rho - 3r^2\rho + 20r^2 - 8r\rho + 32r - 4\rho + 16)}{(2r+1)(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}y_{n+1} \\ &+ \frac{4(r^2 + 3r + 2)}{(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}hf_{n+2} + \frac{4(r^2 + 3r + 2)}{(2r^2\rho + 6r^2 + 3r\rho + 24r + \rho + 20)}h\rho f_{n+1} \end{aligned} \quad (12)$$

For stability considerations, the free parameter ρ is chosen within the interval $(-1, 1)$ as in [5,6,10,17,18,19,23,26]. In this study, the value $\rho = -\frac{1}{4}$ is adopted. By substituting $r = 1$, $r = 2$ and $r = \frac{5}{8}$ into equations (8) and (12), the corresponding coefficients of the proposed method are obtained and subsequently stored for use in the computational implementation. These specific values of r are selected to ensure that the MVSBBDF method preserves its zero-stability property:

For $r = 1$

$$\left. \begin{aligned} y_{n+1} &= \frac{5}{32}y_{n-2} - y_{n-1} + \frac{9}{4}y_n - \frac{13}{32}y_{n+2} + \frac{3}{2}hf_{n+1} - \frac{3}{8}hf_n \\ y_{n+2} &= -\frac{13}{97}y_{n-2} + \frac{70}{97}y_{n-1} - \frac{162}{97}y_n + \frac{202}{97}y_{n+1} + \frac{48}{97}hf_{n+2} - \frac{12}{97}hf_{n+1} \end{aligned} \right\} \quad (13)$$

For $r = 2$

$$\left. \begin{aligned} y_{n+1} &= \frac{1}{16}y_{n-2} - \frac{35}{64}y_{n-1} + \frac{45}{16}y_n - \frac{85}{64}y_{n+2} + \frac{15}{4}hf_{n+1} - \frac{15}{16}hf_n \\ y_{n+2} &= -\frac{7}{353}y_{n-2} + \frac{53}{353}y_{n-1} - \frac{333}{353}y_n + \frac{640}{353}y_{n+1} + \frac{192}{353}hf_{n+2} - \frac{48}{353}hf_{n+1} \end{aligned} \right\} \quad (14)$$

For $r = \frac{5}{8}$

$$\left. \begin{aligned} y_{n+1} &= \frac{1984}{5575}y_{n-2} - \frac{70656}{26075}y_{n-1} + \frac{60021}{22300}y_n - \frac{1479}{6244}y_{n+2} + \frac{234}{223}hf_{n+1} - \frac{117}{446}hf_n \\ y_{n+2} &= -\frac{162176}{349725}y_{n-2} + \frac{231424}{116575}y_{n-1} - \frac{330057}{116575}y_n + \frac{32312}{13989}y_{n+1} + \frac{2184}{4663}hf_{n+2} - \frac{546}{4663}hf_{n+1} \end{aligned} \right\} \quad (15)$$

The derived coefficients in equations (13)-(15) correspond to the different step-size ratios $r = 1$, $r = 2$ and $r = \frac{5}{8}$, representing constant, halved, and increased step sizes respectively. These parameter choices are carefully selected to maintain the zero-stability and convergence of the proposed Modified Variable Step-Size Superclass Block BDF (MVSBBDF) method while ensuring flexibility in adaptive step-size control. Furthermore, the consistency of the coefficients across varying step-size ratios indicates that the method remains robust and stable under moderate step-size variations, which is essential for the efficient numerical integration of stiff and oscillatory IVPs.

Since the proposed method functions as a corrector-type scheme that can be implemented within the PECE (Predict-Evaluate-Correct-Evaluate) framework, it is necessary to derive a corresponding predictor formula. The subsequent subsection presents the derivation of an explicit predictor expression that aligns with the block structure and order of the MVSBBDF method, providing suitable initial approximations for the corrector stage and enhancing the overall computational efficiency of the algorithm.

2.2 Derivation of Predictor Formulae

The initial approximation is predicted using the explicit variable step-size predictor (EVSP) method, which is derived by applying Taylor's expansion to the following formula:

$$\sum_{j=0}^4 \sigma_{j,i,r} y_{n+j-2} = 0 \quad k = i = 1, 2. \quad (16)$$

The derivation of equation (16) involves setting the coefficient $\beta_{k,i,r} = 0$ in the right-hand side of the general K-step linear multistep method (LMM). To determine the coefficient of the first and second points for the predictor method (16), we introduce the linear operator associated with (16) as:

$$L_i[y(x_n), h] = \sigma_{0,i} y(x_n - 2rh) + \sigma_{1,i} y(x_n - rh) + \sigma_{2,i} y(x_n) + \sigma_{3,i} y(x_n + h) + \sigma_{4,i} y(x_n + 2h) = 0, \quad (17)$$

First Point: $k = i = 1$

By setting the coefficient $\sigma_{4,1} = 0$ in (17), the linear operator (17) becomes:

$$L_1[y(x_n), h] = \sigma_{0,1} y(x_n - 2rh) + \sigma_{1,1} y(x_n - rh) + \sigma_{2,1} y(x_n) + \sigma_{3,1} y(x_n + h) = 0, \quad (18)$$

By expanding equation (18) in a Taylor series about the point x_n , the following system of simultaneous linear equations is obtained:

$$\left. \begin{aligned} \epsilon_{0,1} &= \sigma_{0,1} + \sigma_{1,1} + \sigma_{2,1} + \sigma_{3,1} = 0 \\ \epsilon_{1,1} &= -2r\sigma_{0,1} - r\sigma_{1,1} + \sigma_{3,1} = 0 \\ \epsilon_{2,1} &= 2r^2\sigma_{0,1} + \frac{1}{2}r^2\sigma_{1,1} + \frac{1}{2}\sigma_{3,1} = 0 \end{aligned} \right\}. \quad (19)$$

By setting $\sigma_{3,1} = 1$ and solving the resulting system of equations using the Maple 18 environment, the coefficients corresponding to the first point were obtained, as presented in Table 1 below:

Table 1. The Coefficient of First Point for Predictor formulae

$\sigma_{0,1}$	$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{3,1}$
$-\frac{1}{2} \frac{r+1}{r^2}$	$\frac{2r+1}{r^2}$	$-\frac{1}{2} \frac{2r^2+3r+1}{r^2}$	1

Upon substituting the obtained coefficients into equation (18), the formula corresponding to the first point is obtained as follows:

$$-\frac{1}{2} \frac{r+1}{r^2} y_{n-2} + \frac{2r+1}{r^2} y_{n-1} - \frac{1}{2} \frac{2r^2+3r+1}{r^2} y_n + y_{n+1} = 0, \quad (20)$$

which is equivalent to

$$y_{n+1} = \frac{1}{2} \frac{r+1}{r^2} y_{n-2} - \frac{2r+1}{r^2} y_{n-1} + \frac{1}{2} \frac{2r^2+3r+1}{r^2} y_n. \quad (21)$$

Second Point: $k = i = 2$

In a similar manner, setting the coefficient $\sigma_{3,2} = 0$ in equation (17) transforms the linear operator into the following form:

$$L_2(y(x_n), h) = \sigma_{0,2} y(x_n - 2rh) + \sigma_{1,2} y(x_n - rh) + \sigma_{2,2} y(x_n) + \sigma_{4,2} y(x_n + 2h) = 0, \quad (22)$$

Expanding (22) into Taylor series about the point x_n yields the following system of simultaneous linear equations:

$$\left. \begin{aligned} \epsilon_{0,2} &= \sigma_{0,2} + \sigma_{1,2} + \sigma_{2,2} + \sigma_{4,2} = 0 \\ \epsilon_{1,2} &= -2r\sigma_{0,2} - r\sigma_{1,2} + \sigma_{4,2} = 0 \\ \epsilon_{2,2} &= 2r^2\sigma_{0,2} + \frac{1}{2}r^2\sigma_{1,2} + \frac{1}{2}\sigma_{4,2} = 0 \end{aligned} \right\}. \quad (23)$$

Setting $\sigma_{4,2} = 1$ and solving the resulting system of equations in Maple 18 environment yields the coefficients for the second point, as presented in Table 2 below:

Table 2. The Coefficient of Second Point for Predictor Formulae

$\sigma_{0,2}$	$\sigma_{1,2}$	$\sigma_{2,2}$	$\sigma_{4,2}$
$-\frac{r+2}{r^2}$	$\frac{4(r+1)}{r^2}$	$-\frac{r^2+3r+2}{r^2}$	1

The substitution of the obtained coefficients into equation (22) results in the following formula for the second point:

$$-\frac{r+1}{r^2} y_{n-2} + \frac{4(r+1)}{r^2} y_{n-1} - \frac{r^2+3r+2}{r^2} y_n + y_{n+2} = 0, \quad (24)$$

which is equivalent to

$$y_{n+2} = \frac{r+1}{r^2} y_{n-2} - \frac{4(r+1)}{r^2} y_{n-1} + \frac{r^2+3r+2}{r^2} y_n. \quad (25)$$

Hence, the 2-point EVSP method is therefore given by

$$\left. \begin{aligned} y_{n+1} &= \frac{1}{2} \frac{r+1}{r^2} y_{n-2} - \frac{2r+1}{r^2} y_{n-1} + \frac{1}{2} \frac{2r^2+3r+1}{r^2} y_n \\ y_{n+2} &= \frac{r+1}{r^2} y_{n-2} - \frac{4(r+1)}{r^2} y_{n-1} + \frac{r^2+3r+2}{r^2} y_n \end{aligned} \right\}. \quad (26)$$

Considering the step-size ratios $r = 1$, $r = 2$ and $r = \frac{5}{8}$, which correspond to constant, halving, and increasing the step size by a factor of 1.6 in equation (26), we obtain:

For $r = 1$

$$\begin{aligned} y_{n+1} &= y_{n-2} - 3y_{n-1} + 3y_n \\ y_{n+2} &= 3y_{n-2} - 8y_{n-1} + 6y_n \end{aligned} \quad (27)$$

For $r = 2$

$$\begin{aligned} y_{n+1} &= \frac{3}{8}y_{n-2} - \frac{5}{4}y_{n-1} + \frac{15}{8}y_n \\ y_{n+2} &= y_{n-2} - 3y_{n-1} + 3y_n \end{aligned} \quad (28)$$

For $r = \frac{5}{8}$

$$\begin{aligned} y_{n+1} &= \frac{52}{25}y_{n-2} - \frac{144}{25}y_{n-1} + \frac{117}{25}y_n \\ y_{n+2} &= \frac{168}{25}y_{n-2} - \frac{416}{25}y_{n-1} + \frac{273}{25}y_n \end{aligned} \quad (29)$$

It is noteworthy that the predictor formulae presented in equations (27)-(29) were previously reported in Ijam et al. [23], ensuring compatibility and reliability when coupled with the proposed MVSBBDF corrector formulation.

2.3 Order and Error Constant of the MVSBBDF Method

This section presents the derivation of the order and error constant of the method corresponding to equations (13), (14), and (15). Initially, the order of the method is examined. For $r = 1$, equation (13) can be rewritten in the following form:

$$\begin{aligned} -\frac{5}{32}y_{n-2} + y_{n-1} - \frac{9}{4}y_n + y_{n+1} + \frac{13}{32}y_{n+2} &= \frac{3}{2}hf_{n+1} - \frac{3}{8}hf_n \\ \frac{13}{97}y_{n-2} - \frac{70}{97}y_{n-1} + \frac{162}{97}y_n - \frac{202}{97}y_{n+1} + y_{n+2} &= \frac{48}{97}hf_{n+2} - \frac{12}{97}hf_{n+1} \end{aligned} \quad (30)$$

The general matrix form of (30) is defined and expressed as follows:

$$\sum_{j=0}^2 A_j^* Y_{m+j-2} = h \sum_{j=0}^2 B_j^* F_{m+j-2}, \quad (31)$$

where A_0^* , A_1^* , A_2^* , B_0^* , B_1^* , and B_2^* are two by two square matrices defined by:

$$A_0^* = \begin{bmatrix} 0 & -\frac{5}{32} \\ 0 & \frac{13}{97} \end{bmatrix}, A_1^* = \begin{bmatrix} 1 & -\frac{9}{4} \\ -\frac{70}{97} & \frac{162}{97} \end{bmatrix}, A_2^* = \begin{bmatrix} 1 & \frac{13}{32} \\ -\frac{202}{97} & 1 \end{bmatrix}, B_0^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_1^* = \begin{bmatrix} 0 & -\frac{3}{8} \\ 0 & 0 \end{bmatrix}, B_2^* = \begin{bmatrix} \frac{3}{2} & 0 \\ -\frac{12}{97} & \frac{48}{97} \end{bmatrix}$$

and Y_{m-2} , Y_{m-1} , Y_m , F_{m-2} , F_{m-1} and F_m are column vectors defined by

$$Y_{m-2} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}, Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}, F_{m-2} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}$$

The matrix formulation of (31) is given as:

$$\begin{aligned} \begin{bmatrix} 0 & -\frac{5}{32} \\ 0 & \frac{13}{97} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix} + \begin{bmatrix} 1 & -\frac{9}{4} \\ -\frac{70}{97} & \frac{162}{97} \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 1 & \frac{13}{32} \\ -\frac{202}{97} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} &= h \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \end{bmatrix} + h \begin{bmatrix} 0 & -\frac{3}{8} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} \\ + h \begin{bmatrix} \frac{3}{2} & 0 \\ -\frac{12}{97} & \frac{48}{97} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} & \end{aligned} \quad (32)$$

Let A_0^* , A_1^* , A_2^* , B_0^* , B_1^* , and B_2^* block matrices defined by:

$$A_0^* = (A_0 \ A_1), A_1^* = (A_2 \ A_3), A_2^* = (A_4 \ A_5),$$

$$B_0^* = (B_0 \ B_1), B_1^* = (B_2 \ B_3), B_2^* = (B_4 \ B_5),$$

where,

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, A_1 = \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix}, A_2 = \begin{bmatrix} 1 \\ -\frac{70}{97} \end{bmatrix}, A_3 = \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix}, A_4 = \begin{bmatrix} 1 \\ -\frac{202}{97} \end{bmatrix}, A_5 = \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} -\frac{3}{8} \\ 0 \end{bmatrix}, \\ B_4 &= \begin{bmatrix} \frac{3}{2} \\ -\frac{12}{97} \end{bmatrix}, B_5 = \begin{bmatrix} 0 \\ \frac{48}{97} \end{bmatrix}, \end{aligned}$$

Definition 2 (Order). The order of the block method (13) and its associated linear difference operator

$$L[y(x); h] = \sum_{j=0}^4 [A_j y(x + jh) - h B_j y'(x + jh)] \quad (33)$$

is defined as the unique integer p such that the difference operator (33) and the associated method (13) are said to be of order p if

$$C_0 = C_1 = C_2 = \dots = C_p = 0 \text{ and } C_{p+1} \neq 0 \text{ [30].}$$

where,

$$C_0 = \sum_{j=0}^5 A_j = A_1 + A_2 + A_3 + A_4 + A_5 = \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{70}{97} \end{bmatrix} + \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{202}{97} \end{bmatrix} + \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
 C_1 &= \sum_{j=0}^5 (JA_j - B_j) = (A_1 + 2A_2 + 3A_3 + 4A_4 + 5A_5) - (B_3 + B_4 + B_5) = \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{70} \\ -\frac{70}{97} \end{bmatrix} + 2 \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix} + \\
 & 3 \begin{bmatrix} \frac{1}{202} \\ -\frac{202}{97} \end{bmatrix} + \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{3}{8} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ -\frac{12}{97} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{48}{97} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 C_2 &= \sum_{j=0}^5 \left(\frac{1}{2!} J^2 A_j - \frac{1}{1!} J B_j \right) = \frac{1}{2!} (A_1 + 2^2 A_2 + 3^2 A_3 + 4^2 A_4 + 5^2 A_5) - \frac{1}{1!} (3B_3 + 4B_4 + 5B_5) = \frac{1}{2!} \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix} + \\
 & 2^2 \begin{bmatrix} \frac{1}{70} \\ -\frac{70}{97} \end{bmatrix} + 3^2 \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix} + 4^2 \begin{bmatrix} \frac{1}{202} \\ -\frac{202}{97} \end{bmatrix} + 5^2 \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix} - \frac{1}{1!} \begin{bmatrix} -\frac{3}{8} \\ 0 \end{bmatrix} + 4 \begin{bmatrix} \frac{3}{2} \\ -\frac{12}{97} \end{bmatrix} + 5 \begin{bmatrix} 0 \\ \frac{48}{97} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 C_3 &= \sum_{j=0}^5 \left(\frac{1}{3!} J^3 A_j - \frac{1}{2!} J^2 B_j \right) = \frac{1}{3!} (A_1 + 2^3 A_2 + 3^3 A_3 + 4^3 A_4 + 5^3 A_5) - \frac{1}{2!} (3^2 B_3 + 4^2 B_4 + 5^2 B_5) = \frac{1}{3!} \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix} + \\
 & 2^3 \begin{bmatrix} \frac{1}{70} \\ -\frac{70}{97} \end{bmatrix} + 3^3 \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix} + 4^3 \begin{bmatrix} \frac{1}{202} \\ -\frac{202}{97} \end{bmatrix} + 5^3 \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix} - \frac{1}{2!} \begin{bmatrix} -\frac{3}{8} \\ 0 \end{bmatrix} + 4^2 \begin{bmatrix} \frac{3}{2} \\ -\frac{12}{97} \end{bmatrix} + 5^2 \begin{bmatrix} 0 \\ \frac{48}{97} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 C_4 &= \sum_{j=0}^5 \left(\frac{1}{4!} J^4 A_j - \frac{1}{3!} J^3 B_j \right) = \frac{1}{4!} (A_1 + 2^4 A_2 + 3^4 A_3 + 4^4 A_4 + 5^4 A_5) - \frac{1}{3!} (3^3 B_3 + 4^3 B_4 + 5^3 B_5) = \frac{1}{4!} \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix} + \\
 & 2^4 \begin{bmatrix} \frac{1}{70} \\ -\frac{70}{97} \end{bmatrix} + 3^4 \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix} + 4^4 \begin{bmatrix} \frac{1}{202} \\ -\frac{202}{97} \end{bmatrix} + 5^4 \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix} - \frac{1}{3!} \begin{bmatrix} -\frac{3}{8} \\ 0 \end{bmatrix} + 4^3 \begin{bmatrix} \frac{3}{2} \\ -\frac{12}{97} \end{bmatrix} + 5^3 \begin{bmatrix} 0 \\ \frac{48}{97} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 C_5 &= \sum_{j=0}^5 \left(\frac{1}{5!} J^5 A_j - \frac{1}{4!} J^4 B_j \right) = \frac{1}{5!} (A_1 + 2^5 A_2 + 3^5 A_3 + 4^5 A_4 + 5^5 A_5) - \frac{1}{4!} (3^4 B_3 + 4^4 B_4 + 5^4 B_5) = \frac{1}{5!} \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix} + \\
 & 2^5 \begin{bmatrix} \frac{1}{70} \\ -\frac{70}{97} \end{bmatrix} + 3^5 \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix} + 4^5 \begin{bmatrix} \frac{1}{202} \\ -\frac{202}{97} \end{bmatrix} + 5^5 \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix} - \frac{1}{4!} \begin{bmatrix} -\frac{3}{8} \\ 0 \end{bmatrix} + 4^4 \begin{bmatrix} \frac{3}{2} \\ -\frac{12}{97} \end{bmatrix} + 5^4 \begin{bmatrix} 0 \\ \frac{48}{97} \end{bmatrix} = \begin{bmatrix} \frac{7}{80} \\ -\frac{51}{485} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

In accordance with the previously stated definition 2, it is demonstrated that the MVSBBDF is derived to be of order four and its corresponding error constant is expressed as:

$$C_5 = \begin{bmatrix} \frac{7}{80} \\ -\frac{51}{485} \end{bmatrix}.$$

Using the same procedure employed to establish the order of method (13), it can be verified that methods (14) and (15) also achieve fourth-order accuracy.

2.4 Stability Analysis of the Method

In this section, the stability characteristics of the MVSBBDF method are examined, with particular focus on zero- and A-stability. This analysis highlights the method's suitability for solving stiff oscillatory problems. For practical applicability in such problems, the MVSBBDF method should ideally exhibit, at a minimum almost A-stability properties.

Definition 3 (Zero-Stability). A linear multistep method (LMM) is said to be zero-stable if all the roots of its first characteristic polynomial lie within or on the unit circle, and any root on the unit circle is simple [31]. This condition guarantees that small perturbations or rounding errors introduced at any step do not grow unboundedly as the computation proceeds, thereby ensuring the method's long-term numerical stability.

Definition 4 (A-Stability). A linear multistep method (LMM) is said to be A-stable if its region of absolute stability contains the entire left half of the complex plane [32].

The linear stability properties of the methods are analyzed using the standard first-order linear test equation. Specifically, the stability of methods (13), (14), and (15) is assessed by applying the Dahlquist linear test equation for first-order ordinary differential equations:

$$y' = \lambda y, \lambda < 0 \quad (34)$$

To examine the stability of the MVSBBDF method for $r = 1$, the substitution of equation (34) into the expressions defined in (13) results in:

$$\left. \begin{aligned} y_{n+1} &= \frac{5}{32}y_{n-2} - y_{n-1} + \frac{9}{4}y_n - \frac{13}{32}y_{n+2} + \frac{3}{2}h\lambda y_{n+1} - \frac{3}{8}h\lambda y_n \\ y_{n+2} &= -\frac{13}{97}y_{n-2} + \frac{70}{97}y_{n-1} - \frac{162}{97}y_n + \frac{202}{97}y_{n+1} + \frac{48}{97}h\lambda y_{n+2} - \frac{12}{97}h\lambda y_{n+1} \end{aligned} \right\} \quad (35)$$

After rearranging and combining like terms, the resulting expression is

$$\left. \begin{aligned} \left(1 - \frac{3}{2}h\lambda\right)y_{n+1} + \frac{13}{32}y_{n+2} &= \frac{5}{32}y_{n-2} - y_{n-1} + \left(\frac{9}{4} - \frac{3}{8}h\lambda\right)y_n \\ \left(-\frac{202}{97} + \frac{12}{97}h\lambda\right)y_{n+1} + \left(1 - \frac{48}{97}h\lambda\right)y_{n+2} &= -\frac{13}{97}y_{n-2} + \frac{70}{97}y_{n-1} - \frac{162}{97}y_n \end{aligned} \right\} \quad (36)$$

The matrix form corresponding to equation (36) can be written as:

$$\begin{bmatrix} \left(1 - \frac{3}{2}h\lambda\right) & \frac{13}{32} \\ \left(-\frac{202}{97} + \frac{12}{97}h\lambda\right) & \left(1 - \frac{48}{97}h\lambda\right) \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} -1 & \left(\frac{9}{4} - \frac{3}{8}h\lambda\right) \\ \frac{70}{97} & -\frac{162}{97} \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{32} \\ 0 & -\frac{13}{97} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix} \quad (37)$$

Letting $h\lambda = \bar{h}$ in (37) gives:

$$\begin{bmatrix} \left(1 - \frac{3}{2}\bar{h}\right) & \frac{13}{32} \\ \left(-\frac{202}{97} + \frac{12}{97}\bar{h}\right) & \left(1 - \frac{48}{97}\bar{h}\right) \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} -1 & \left(\frac{9}{4} - \frac{3}{8}\bar{h}\right) \\ \frac{70}{97} & -\frac{162}{97} \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{32} \\ 0 & -\frac{13}{97} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix} \quad (38)$$

Definition 5. If m is the number of block and r is the number of points in the block, then $n = mr$. Here $r = 2$ and $n = 2m$. By [33], we let

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} y_{2m+1} \\ y_{2m+2} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{2m-1} \\ y_{2m} \end{bmatrix} = \begin{bmatrix} y_{2(m-1)+1} \\ y_{2(m-1)+2} \end{bmatrix}, Y_{m-2} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix} = \begin{bmatrix} y_{2(m-2)+1} \\ y_{2(m-2)+2} \end{bmatrix},$$

Equation (38) is equivalently written as

$$DY_m = EY_{m-1} + FY_{m-2} \quad (39)$$

where

$$D = \begin{bmatrix} \left(1 - \frac{3}{2}\bar{h}\right) & \frac{13}{32} \\ \left(-\frac{202}{97} + \frac{12}{97}\bar{h}\right) & \left(1 - \frac{48}{97}\bar{h}\right) \end{bmatrix}, E = \begin{bmatrix} -1 & \left(\frac{9}{4} - \frac{3}{8}\bar{h}\right) \\ \frac{70}{97} & -\frac{162}{97} \end{bmatrix}, F = \begin{bmatrix} 0 & \frac{5}{32} \\ 0 & -\frac{13}{97} \end{bmatrix}.$$

By evaluating the determinant of the quadratic equation $Dt^2 - Et - F = 0$, the stability polynomial $\psi_1(t, \bar{h})$ is obtained as:

$$\psi_1(t, \bar{h}) = -\frac{225}{1552}t^2 + \frac{33}{1552}t + \frac{69}{776}\bar{h}t^2 - \frac{9}{194}\bar{h}^2t^3 + \frac{72}{97}\bar{h}^2t^4 - \frac{1587}{776}\bar{h}t^4 - \frac{753}{388}\bar{h}t^3 - \frac{2673}{1552}t^3 + \frac{2865}{1552}t^4 = 0. \quad (40)$$

The zero-stability of the method is verified by substituting $\bar{h} = 0$ into equation (40), which gives the following first characteristic polynomial:

$$-\frac{225}{1552}t^2 + \frac{33}{1552}t - \frac{2673}{1552}t^3 + \frac{2865}{1552}t^4 = 0. \quad (41)$$

By solving equation (41) with respect to t , the following roots are obtained:

$$t = 0, t = 1, t = -0.1459405037, t = 0.07892479690.$$

These root values confirm that the method is zero-stable, since no root has magnitude greater than one and the root $t = 1$ is simple and unique.

Applying the same procedure, the stability polynomial of method (14) for $r = 2$ is given by:

$$\psi_2(t, \bar{h}) = \frac{3}{22592}t^2 + \frac{33}{22592}t + \frac{423}{5648}\bar{h}t^2 - \frac{45}{353}\bar{h}^2t^3 + \frac{720}{353}\bar{h}^2t^4 - \frac{3159}{706}\bar{h}t^4 - \frac{2475}{1412}\bar{h}t^3 - \frac{19257}{5648}t^3 + \frac{1203}{353}t^4 = 0. \quad (42)$$

substituting $\bar{h} = 0$ into equation (42), the first characteristics polynomial of the method (14) becomes:

$$\frac{3}{22592}t^2 + \frac{33}{22592}t - \frac{19257}{5648}t^3 + \frac{1203}{353}t^4 = 0. \quad (43)$$

The roots of equation (43) with respect to t are given by:

$$t = 0, t = 1, t = -0.020470571178, t = 0.02093815282.$$

Hence, the method (14) is zero-stable.

Similarly, the stability polynomial of method (15) associated with $r = \frac{5}{8}$ is expressed as

$$\begin{aligned} \psi_{\frac{5}{8}}(t, \bar{h}) &= -\frac{74784384}{129981125}t^2 + \frac{17301504}{129981125}t + \frac{394368}{5199245}\bar{h}t^2 - \frac{31941}{1039849}\bar{h}^2t^3 + \frac{511056}{1039849}\bar{h}^2t^4 - \frac{3214029}{2079698}\bar{h}t^4 - \frac{150666087}{51992450}\bar{h}t^3 \\ &\quad - \frac{28722699}{25996225}t^3 + \frac{1608771}{1039849}t^4 = 0. \end{aligned} \quad (44)$$

The first characteristics polynomial of the method is determined by substituting $\bar{h} = 0$ in (44) to obtain

$$-\frac{74784384}{129981125}t^2 + \frac{17301504}{129981125}t - \frac{28722699}{25996225}t^3 + \frac{1608771}{1039849}t^4 = 0. \quad (45)$$

By solving the above fourth degree polynomial for t , we have the following roots

$t = 0, t = 1, t = -0.4692104995, t = 0.1833630793$.

Again, it is shown that the method is zero-stable since no magnitude of the roots is greater than one and the root $t = 1$ is simple. The absolute stability region of the MVSBBDF method is analyzed in the $h\lambda$ complex plane. The boundary of the stability region is defined by the locus of points $t = e^{i\theta} = \cos \theta + i \sin \theta, \quad 0 \leq \theta \leq 2\pi$. (46)

To determine this boundary, the root condition of the corresponding stability polynomial is evaluated over a suitably fine grid in the stability space. The absolute stability region is then identified as the set of all points satisfying $|t| < 1$. The resulting stability region associated with MVSBBDF method for $r = 1, r = 2$ and $r = 5/8$ is illustrated below, as generated using the Maple 18 computation environment.

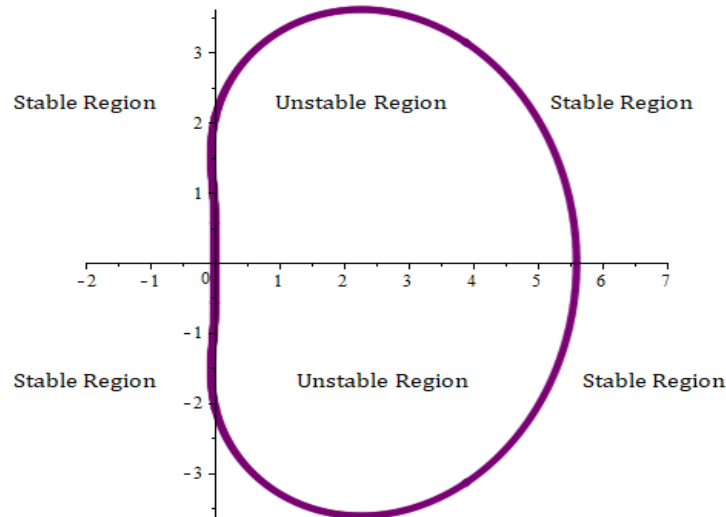


Figure 2. Stability Region for MVSBBDF method when $r = 1$

Thus, the region of absolute stability of the stability polynomial (40) lies outside the circle, indicating that method (13) is A-stable, as the stability region covers the entire left half of the complex plane.

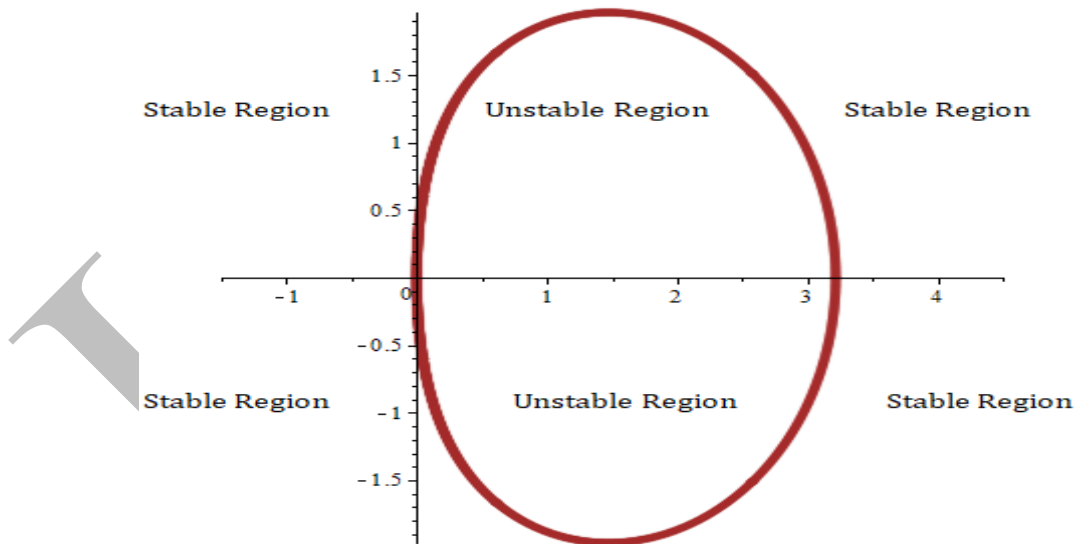


Figure 3. Stability Region for MVSBBDF method when $r = 2$

The stability plot of polynomial (42), presented in Figure 3, is also A-stable because its absolute stability region is exterior to the circle and encompasses the whole left half-plane.

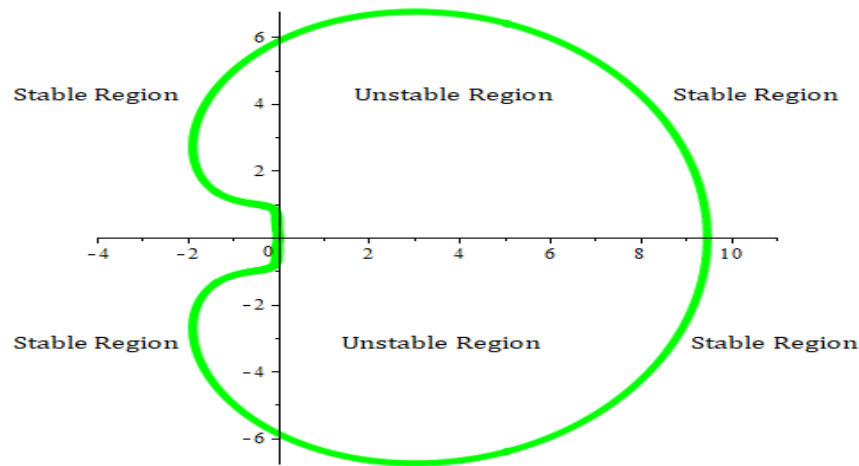


Figure 4. Stability Region for MVSBBDF method when $r = 5/8$

It is observed that the absolute stability region lies outside the circle, indicating that equation (15) is nearly A-stable, as the region covers almost the entire left half of the complex plane. A-stability is one of the key properties desired for the numerical solution of stiff problems. Therefore, the developed methods are suitable for the numerical integration of stiff and oscillatory differential equations. Figure 5 provides a visual representation of the MVSBBDF method for $r = 1$, $r = 2$ and $r = 5/8$, allowing a direct comparison of their respective stability regions

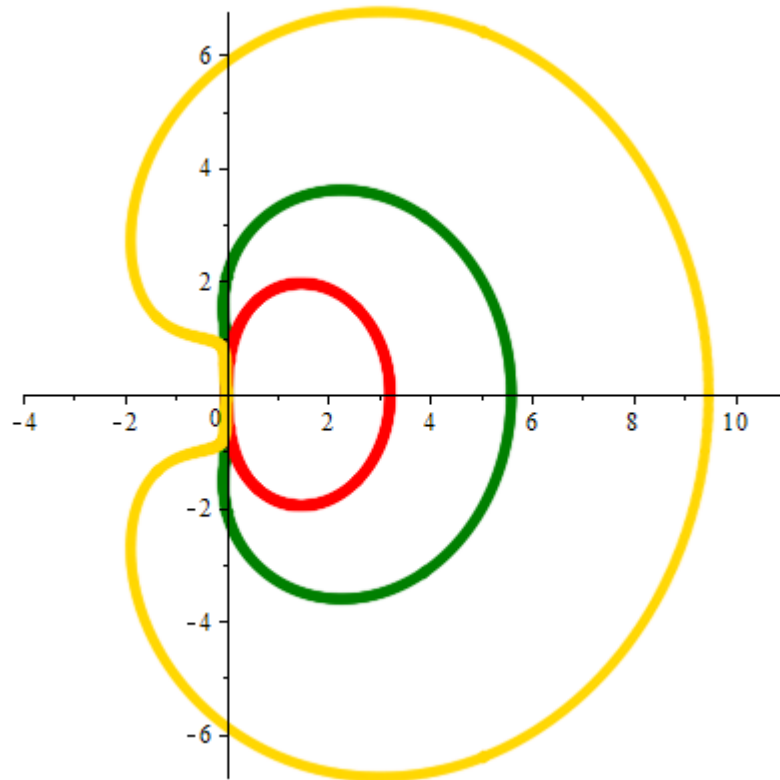


Figure 5. Comparison of the stability region for MVSBBDF method when $r = 1$, $r = 2$ and $r = 5/8$

Table 1 provides a comprehensive comparison of the methods depicted in Figure 5, detailing their absolute stability characteristics along with the corresponding intervals of instability.

Table 1. Comparison of absolute stability and Interval of instability

Step-Size Ratio	Interval of Instability	Colour of the Graph	Status of the Method
$r = 1$	$(-0.056, 4.6)$	Green	Absolutely Stable
$r = 2$	$(0, 3.218)$	Red	Absolutely Stable
$r = 5/8$	$(-1.884, 9.479)$	Yellow	Absolutely Stable

The stability region of the MVSBBDF method depends on the step-size ratio, as the coefficients of the adaptive step-size method vary with r . When r becomes very small, the large variation between consecutive step sizes significantly alters the stability polynomial, potentially reducing the absolute stability region and enlarging the unstable region. The data in Table 1 indicate that the MVSBBDF method for $r = 2$ has the narrowest interval of instability, corresponding to the widest stability region. Conversely, for $r = 5/8$, the method exhibits the largest interval of instability, implying the smallest stability region.

2.5 Convergence of the Method

The MVSBBDF method must be convergent before it can be considered suitable for the numerical approximation of any stiff system, as a method that does not converge lacks practical utility. This section is dedicated to establishing the necessary and sufficient conditions for the convergence of the MVSBBDF method. According to [34], consistency and zero-stability are both necessary and sufficient for the convergence of any linear multistep scheme. To demonstrate that the method (13) satisfies these criteria, we first present definition of consistency and a Dahlquist equivalence theorem pertaining to the convergence of linear multistep methods, followed by a detailed verification of its consistency and zero-stability properties.

Definition 6 (Consistency): A linear multistep method (LMM) is said to be consistent if it has order $p \geq 1$. It follows that a LMM is consistent if the following conditions are satisfied:

- i. $\sum_{j=0}^5 A_j = 0$,
- ii. $\sum_{j=0}^5 jA_j = \sum_{j=0}^5 B_j$.

Based on definition 2, from section 2.3, we deduced that the order of the MVSBBDF method in (13) is four which is greater than 1. Hence, in accordance with the definition 6, the method is consistent.

Let $A_0, A_1, A_2, A_3, A_4, A_5, B_0, B_1, B_2, B_3, B_4$, and B_5 be column matrices as previously defined by, then

$$\sum_{j=0}^5 A_j = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix} + \begin{bmatrix} \frac{1}{70} \\ -\frac{70}{97} \end{bmatrix} + \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix} + \begin{bmatrix} \frac{1}{202} \\ -\frac{202}{97} \end{bmatrix} + \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, the first consistency condition in (i) is satisfied.

Similarly,

$$\sum_{j=0}^5 jA_j = (0A_0 + 1A_1 + 2A_2 + 3A_3 + 4A_4 + 5A_5) = \begin{bmatrix} 1 \begin{bmatrix} -\frac{5}{32} \\ \frac{13}{97} \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{70} \\ -\frac{70}{97} \end{bmatrix} + 2 \begin{bmatrix} -\frac{9}{4} \\ \frac{162}{97} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{202} \\ -\frac{202}{97} \end{bmatrix} + \begin{bmatrix} \frac{13}{32} \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \frac{9}{8} \\ \frac{36}{97} \end{bmatrix}.$$

$$\sum_{j=0}^5 B_j = (B_0 + B_1 + B_2 + B_3 + B_4 + B_5) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3}{8} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ -\frac{12}{97} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{48}{97} \end{bmatrix} = \begin{bmatrix} \frac{9}{8} \\ \frac{36}{97} \end{bmatrix}.$$

Hence, it is clear that $\sum_{j=0}^5 jA_j = \sum_{j=0}^5 B_j$.

Therefore, the second consistency condition in (ii) is satisfied, indicating that all required consistency criteria are fulfilled. Consequently, the MVSBBDF method is confirmed to be consistent.

Theorem 1. A linear multistep method (LMM) is convergent if and only if it is both consistent and zero-stable.

Theorem 2. The MVSBBDF method is convergent

Proof:

From the preceding analysis, the MVSBBDF method is of order four, which satisfies the condition for consistency. Additionally, the roots of the first characteristic polynomial of the method lie within or on the unit circle, with any root on the unit circle being simple, confirming that the method is zero-stable for all considered step-size ratios $r = 1, r = 2$ and $r = 5/8$.

Since the MVSBBDF method satisfies both consistency and zero-stability, it meets the necessary and sufficient criteria for convergence. Therefore, the method is convergent, ensuring that the numerical solution produced by the MVSBBDF method reliably approximates the exact solution of stiff and oscillatory initial value problems as the step size decreases.

2.6 Implementation of the Method

Newton's iteration serves as the basis for implementing the method. The procedure is outlined below, with formulae (13)-(15) expressed in the following form:

$$\left. \begin{aligned} y_{n+1} &= \theta_1 y_{n+2} + \alpha_1 h f_n + \alpha_2 h f_{n+1} + \tau_1 \\ y_{n+2} &= \theta_2 y_{n+1} + \alpha_3 h f_{n+1} + \alpha_4 h f_{n+2} + \tau_2 \end{aligned} \right\} \quad (47)$$

where τ_1 and τ_2 are the back values.

Equation (47) can be reformulated as:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} + h \begin{bmatrix} 0 & \alpha_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} \alpha_2 & 0 \\ \alpha_3 & \alpha_4 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}. \quad (48)$$

which can be represented as

$$(I - G)Y = h(H_1 F_1 + H_2 F_2) + \Psi, \quad (49)$$

where,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{bmatrix}, H_1 = \begin{bmatrix} 0 & \alpha_1 \\ 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} \alpha_2 & 0 \\ \alpha_3 & \alpha_4 \end{bmatrix}, \psi = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix},$$

$$\psi = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}, Y = \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix}, F_1 = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}, F_2 = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}.$$

Let

$$K = (I - G)Y - h(H_1 F_1 + H_2 F_2) - \Psi = 0, \quad (50)$$

Newton's iteration applied to the MVSBBDF method is expressed as:

$$Y_{n+1,n+2}^{(i+1)} - Y_{n+1,n+2}^{(i)} = - \left(K'_j(y_{n+1,n+2}^{(i)}) \right)^{-1} \left(K_j(y_{n+1,n+2}^{(i)}) \right), \quad (51)$$

Equation (51) is equivalent to

$$Y_{n+1,n+2}^{(i+1)} - Y_{n+1,n+2}^{(i)} = - \left((I - G) - hH_1 \frac{\partial F_1}{\partial Y}(y_{n+1,n+2}^{(i)}) - hH_2 \frac{\partial F_2}{\partial Y}(y_{n+1,n+2}^{(i)}) \right)^{-1} \\ \left((I - G)(y_{n+1,n+2}^{(i)}) - hH_1 F_1 - hH_2 F_2 - \Psi \right). \quad (52)$$

Let $Y_{n+1}^{(i+1)}$ denote the $(i + 1)$ th iterate and

$$L_{1,2}^{(i+1)} = Y_{n+1,n+2}^{(i+1)} - Y_{n+1,n+2}^{(i)}. \quad (53)$$

The equation (53) can be written as

$$L_{1,2}^{(i+1)} = \bar{A}^{-1} \bar{B} \quad (54)$$

which equivalent to

$$\bar{A} L_{1,2}^{(i+1)} = \bar{B}, \quad (55)$$

where

$$\bar{A} = \left((I - G) - hH_1 \frac{\partial F_1}{\partial Y}(y_{n+1,n+2}^{(i)}) - hH_2 \frac{\partial F_2}{\partial Y}(y_{n+1,n+2}^{(i)}) \right)^{-1}$$

and

$$\bar{B} = - \left((I - G)(y_{n+1,n+2}^{(i)}) - hH_1 F_1 - hH_2 F_2 - \Psi \right)$$

Consequently, system (55) is solved using Newton's iteration for each specified value of r .

$$\bar{A} = \begin{pmatrix} 1 - \alpha_2 h \frac{\partial f_{n+1}}{\partial y_{n+1}} & -\theta_1 \\ -\theta_2 - \theta_3 h \frac{\partial f_{n+1}}{\partial y_{n+1}} & 1 - \alpha_4 h \frac{\partial f_{n+2}}{\partial y_{n+2}} \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} -y_{n+1}^i + \theta_1 y_{n+2}^i + \alpha_1 h f_n^i + \alpha_2 h f_{n+2}^i + \Psi \\ -y_{n+2}^i + \theta_2 y_{n+1}^i + \alpha_3 h f_{n+1}^i + \alpha_4 h f_{n+2}^i + \Psi \end{pmatrix}.$$

For $r = 1$

$$\bar{A} = \begin{pmatrix} 1 - \frac{3}{2} h \frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{13}{32} \\ -\frac{202}{97} + \frac{12}{97} h \frac{\partial f_{n+1}}{\partial y_{n+1}} & 1 - \frac{192}{353} h \frac{\partial f_{n+2}}{\partial y_{n+2}} \end{pmatrix}, \bar{B} =$$

$$\begin{pmatrix} -y_{n+1}^i - \frac{13}{32} y_{n+2}^i - \frac{3}{8} h f_n^i + \frac{3}{2} h f_{n+2}^i + \frac{5}{32} y_{n-2} - y_{n-1} + \frac{9}{4} y_n \\ -y_{n+2}^i + \frac{86}{53} y_{n+1}^i + \frac{12}{53} h f_{n+1}^i + \frac{24}{53} h f_{n+2}^i - \frac{13}{97} y_{n-2} + \frac{70}{97} y_{n-1} - \frac{162}{97} y_n \end{pmatrix}.$$

For $r = 2$

$$\bar{A} = \begin{pmatrix} 1 + \frac{15}{4} h \frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{85}{64} \\ -\frac{640}{353} + \frac{48}{353} h \frac{\partial f_{n+1}}{\partial y_{n+1}} & 1 - \frac{192}{353} h \frac{\partial f_{n+2}}{\partial y_{n+2}} \end{pmatrix},$$

$$\bar{B} = \begin{pmatrix} -y_{n+1}^i - \frac{85}{64} y_{n+2}^i - \frac{15}{16} h f_n^i + \frac{15}{4} h f_{n+1}^i + \frac{1}{16} y_{n-2} - \frac{35}{64} y_{n-1} + \frac{45}{16} y_n \\ -y_{n+2}^i + \frac{640}{353} y_{n+1}^i - \frac{48}{353} h f_{n+1}^i + \frac{192}{353} h f_{n+2}^i - \frac{7}{353} y_{n-2} + \frac{53}{353} y_{n-1} - \frac{333}{353} y_n \end{pmatrix}.$$

For $r = \frac{5}{8}$

$$\bar{A} = \begin{pmatrix} 1 - \frac{234}{223} h \frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{1479}{6244} \\ -\frac{32312}{13989} + \frac{546}{4663} h \frac{\partial f_{n+1}}{\partial y_{n+1}} & 1 - \frac{2184}{4663} h \frac{\partial f_{n+2}}{\partial y_{n+2}} \end{pmatrix}, \quad \bar{B} =$$

$$\begin{pmatrix} -y_{n+1}^i - \frac{1479}{6244} y_{n+2}^i - \frac{117}{446} h f_n^i + \frac{234}{223} h f_{n+1}^i + \frac{1984}{5575} y_{n-2} - \frac{70656}{26075} y_{n-1} + \frac{60021}{22300} y_n \\ -y_{n+2}^i + \frac{32312}{13989} y_{n+1}^i - \frac{546}{4663} h f_{n+1}^i + \frac{2184}{4663} h f_{n+2}^i - \frac{162176}{349725} y_{n-2} + \frac{231424}{116575} y_{n-1} - \frac{330057}{116575} y_n \end{pmatrix}.$$

A computer algorithm written in the C- programming language is designed to implement the methods.

2.7 Estimating the Local Truncation Error (LTE) and Maximum absolute error (MAXE) of the Method

When implementing an algorithm with finite-precision computations, error analysis is essential to assess the effectiveness of a numerical method. The MVSBBDF method is no exception, as it introduces approximation, rounding, and truncation errors each time it is applied. In this section, the concepts of local truncation error and maximum error are presented. The local truncation error is defined as the difference between two consecutive formulas of orders $p + 1$ and p .

$$LTE = |y_{n+2}^{(p+1)} - y_{n+2}^{(p)}|, \quad (56)$$

For the MVSBBDF Method, the LTEs are given by

For $r = 1$

$$\left. \begin{aligned} y_{n+2}^{(4)} &= -\frac{13}{97}y_{n-2} + \frac{70}{97}y_{n-1} - \frac{162}{97}y_n + \frac{202}{97}y_{n+1} + \frac{48}{97}hf_{n+2} - \frac{12}{97}hf_{n+1} \\ y_{n+2}^{(3)} &= \frac{3}{14}y_{n-1} - y_n + \frac{25}{14}y_{n+1} + \frac{4}{7}hf_{n+2} - \frac{1}{7}hf_{n+1} \\ LTE_1 &= -\frac{52}{679}hf_{n+2} + \frac{13}{679}hf_{n+1} + \frac{403}{1358}y_{n+1} - \frac{13}{97}y_{n-2} + \frac{689}{1358}y_{n-1} - \frac{65}{97}y_n \end{aligned} \right\}. \quad (57)$$

For $r = 2$

$$\left. \begin{aligned} y_{n+2}^{(4)} &= -\frac{7}{353}y_{n-2} + \frac{53}{353}y_{n-1} - \frac{333}{353}y_n + \frac{640}{353}y_{n+1} + \frac{192}{353}hf_{n+2} - \frac{48}{353}hf_{n+1} \\ y_{n+2}^{(3)} &= \frac{3}{53}y_{n-1} - \frac{38}{53}y_n + \frac{88}{53}y_{n+1} + \frac{32}{53}hf_{n+2} - \frac{8}{53}hf_{n+1} \\ LTE_2 &= -\frac{1120}{18709}hf_{n+2} + \frac{280}{18709}hf_{n+1} + \frac{2856}{18709}y_{n+1} - \frac{7}{353}y_{n-2} + \frac{1750}{18709}y_{n-1} - \frac{4235}{18709}y_n \end{aligned} \right\}. \quad (58)$$

For $r = \frac{5}{8}$

$$\left. \begin{aligned} y_{n+2}^{(4)} &= -\frac{162176}{349725}y_{n-2} + \frac{231424}{116575}y_{n-1} - \frac{330057}{116575}y_n + \frac{32312}{13989}y_{n+1} + \frac{2184}{4663}hf_{n+2} - \frac{546}{4663}hf_{n+1} \\ y_{n+2}^{(3)} &= \frac{3072}{6565}y_{n-1} - \frac{679}{505}y_n + \frac{2464}{1313}y_{n+1} + \frac{56}{101}hf_{n+2} - \frac{14}{101}hf_{n+1} \\ LTE_{\frac{5}{8}} &= -\frac{40544}{470963}hf_{n+2} + \frac{10136}{470963}hf_{n+1} + \frac{7956760}{18367557}y_{n+1} - \frac{162176}{349725}y_{n-2} + \frac{232236032}{153062975}y_{n-1} - \frac{17504872}{11774075}y_n \end{aligned} \right\}. \quad (59)$$

To estimate the maximum absolute error of the method, we define the absolute error, as the difference between approximate and exact solution in absolute value which is given by:

$$(\hat{e}_i)_t = |(\hat{y}_i)_t - (y_i)_t|. \quad (60)$$

where $(\hat{y}_i)_t$ is the t^{th} - component of the approximate y .

Then, the maximum error is defined by:

$$MAXE = \max_{1 \leq i \leq s} \left(\max_{1 \leq t \leq s} ((\hat{e}_i)_t) \right). \quad (61)$$

2.8 Step Size Selection Strategy

In the implementation of the MVSBBDF method, the step size is adjusted adaptively based on the success or failure of each integration step. Two possibilities exist for adjusting the step size following a successful step, and one possibility applies in the case of a failed step.

Accordingly, the step size ratio r may take one of the following values:

- $r = 1$: the step size remains unchanged,
- $r = 2$: the step size is halved,
- $r = 5/8$: the step size is increased by a factor of 1.6.

For each successful step, the step size is either maintained ($r = 1$) or increased by the factor 1.6 ($r = 5/8$). Conversely, for a failed step, the step size is reduced to half its current value ($r = 2$).

The approximations y_{n+1} and y_{n+2} are accepted if the current step is successful; otherwise, they are rejected. The success or failure of a step is determined by comparing the local truncation error (LTE) with the user-defined tolerance (TOL). Specifically, if $LTE < TOL$, the step is considered successful; otherwise, it is deemed a failure. The selected test problems are solved using error tolerance limit of 10^{-2} , 10^{-4} and 10^{-6} .

Basically, there are two possibilities in choosing the appropriate step size.

Case 1: Successful step ($LTE < TOL$)

In this case, the computed values y_{n+1} and y_{n+2} are accepted. There are two possibilities for the next step size:

- Maintain the current step size ($r = 1$), if the previous step is failed, or
- Increase the step size by a factor of 1.6 ($r = 5/8$), if the previous step was successful.

The step size increment is computed using

$$h_{new} = c \times h_{old} \times \left(\frac{TOL}{LTE} \right)^{\frac{1}{p}}. \quad (62)$$

and if $h_{new} > 1.6 \times h_{old}$ then $h_{new} = 1.6 \times h_{old}$.

where c is safety factor which is set to be 0.8, p is the order of the method, h_{old} is previous step size and h_{new} is the current step size.

Case 2: Failure step ($LTE > TOL$)

In this situation, the approximations y_{n+1} and y_{n+2} are rejected, and the current step is repeated with a reduced step size:

$$h_{new} = \frac{1}{2} h_{old}. \quad (63)$$

This reduction enhances the accuracy of the next iteration and ensures the error remains within the specified tolerance limits.

3. Test Problems and Numerical Results

The effectiveness and efficiency of the MVSBBDF method are assessed by applying it to the following first-order linear and nonlinear systems exhibiting stiff and oscillatory behavior:

Problem 1: This oscillatory problem is taken from [26, 36]:

$$\begin{aligned} y_1' &= -21y_1 + 19y_2 - 20y_3, & y_1(0) &= 1, & 0 \leq x \leq 10, \\ y_2' &= 19y_1 - 21y_2 + 20y_3, & y_2(0) &= 0, \\ y_3' &= 40y_1 - 40y_2 - 40y_3, & y_3(0) &= -1, \end{aligned}$$

The analytical (exact) solution is given by

$$y_1(x) = 0.5(e^{-2x} + e^{-40x}(\cos 40x + \sin 40x)),$$

$$y_2(x) = 0.5(e^{-2x} - e^{-40x}(\cos 40x + \sin 40x)),$$

$$y_3(x) = 2e^{-40x} \left(-\frac{1}{2} \cos 40x + \frac{1}{2} \sin 40x \right).$$

The eigenvalues of the coefficient matrix of the system are $\lambda_1 = -2$, $\lambda_2 = -40 + 40i$ and $\lambda_3 = -40 - 40i$.

Problem 2: This problem considers a well-known nonlinear stiff reaction kinetics system as presented in [23,26,36]

$$\begin{aligned} y_1' &= -(\epsilon^{-1} + 2)y_1 - \epsilon^{-1} y_2^2, & y_1(0) &= 1, & 0 \leq x \leq 20, \\ y_2' &= y_1 - y_2(1 + y_2), & y_2(0) &= 1, \end{aligned}$$

The analytical (exact) solution is independent of ϵ and is given by

$$y_1(x) = e^{-2x},$$

$$y_2(x) = e^{-x}.$$

The eigenvalues of the Jacobian matrix of the system are $\lambda_1 = -1$ and $\lambda_2 = -(\epsilon^{-1} + 2)$, which indicates that the degree of stiffness in the system can be controlled through the parameter $\epsilon = 10^{-5}$.

Problem 3: The second order non-normal oscillator problem is taken from [26]

$$y''(x) + 2 \in y'(x) + \epsilon^2 y(x) = \epsilon^2 f(x),$$

Subject to the following initial conditions

$$y(0) = \frac{\epsilon^2(\epsilon-1)}{(\epsilon^2+1)^2}, \quad y'(0) = \frac{2\epsilon^3}{(\epsilon^2+1)^2}$$

This differential equation can be rewritten as the following first-order linear system:

$$y_1' = y_2, \quad y_1(0) = \frac{\epsilon^2(\epsilon-1)}{(\epsilon^2+1)^2}, \quad 0 \leq x \leq \pi,$$

$$y_2' = \epsilon^2 \cos(x) - 2 \in y_2 - \epsilon^2 y_1, \quad y_2(0) = \frac{2\epsilon^3}{(\epsilon^2+1)^2}.$$

The exact solution of the system is given by:

$$y_1(x) = e^{\epsilon x},$$

$$y_2(x) = x e^{\epsilon x}.$$

The eigenvalues of the Jacobian matrix are $\lambda_1 = -\epsilon$ and $\lambda_2 = -\epsilon$ confirming the system's stiff behaviour. The stiffness is controlled by the parameter $\epsilon = 10^3$.

Problem 4: The Oscillatory stiff cosine problem, adopted from [23,26,36,37] is given by

$$y' = -2\pi \sin(2\pi x) - \frac{1}{\epsilon} (y - \cos(2\pi x)), \quad y(0) = 1, \quad 0 \leq x \leq 10.$$

The problem exhibits increasing stiffness as $\epsilon \rightarrow 0$. For this test, the stiffness parameter is chosen as $\epsilon = 10^{-3}$. The exact solution is

$$y(x) = \cos(2\pi x).$$

The previously defined test problems were solved using four distinct numerical approaches: the Non-Block Backward Differentiation Formula (NBDF) [8], the fourth order 2-Point Variable Step-Size Block Backward Differentiation Formula (VSBBDF) [24,25], the fourth order 2-Point Variable Step-Size Superclass of Block Backward Differentiation Formula (NVSBBDF) [26], and the Modified Variable Step-Size Superclass of Block Backward Differentiation Formula (MVSBBDF).

The accuracy and computational efficiency of the methods are evaluated and compared using the maximum absolute errors corresponding to different tolerance values (TOL) for each test problem. Furthermore, the numbers of failed steps, successful steps, and total integration steps used in solving each problem are also reported.

Table 2. The notations used in the subsequent tables

SYMBOL	DESCRIPTION
TOL	Tolerance value Used
METHOD	Numerical method used
TRS	Number of failed steps
TAS	Number of successful steps
TS	Total number of steps
MAXE	Maximum absolute error
NBDF	Non-Block Backward Differentiation Formula [8]
VSBBDF	Fourth Order 2-Point Variable Step-Size Block Backward Differentiation Formula [24,25]
NVSBBDF	Fourth order 2-Point Variable Step-Size Superclass of Block Backward Differentiation Formula [26]
MVSBBDF	Modified 2-Point Variable Step-Size Superclass of Block Backward Differentiation Formula

Table 3. Numerical Result for Problem 1

TOL	METHOD	TRS	TAS	TS	MAXE
10^{-2}	NBDF	8	48	56	4.92270E-002
	VSBBDF	0	46	46	4.09523E-003
	NVSBBDF	0	48	48	5.56533E-005
	MVSBBDF	0	49	49	1.46981E-005
10^{-4}	NBDF	15	100	115	5.96300E-004
	VSBBDF	0	66	66	4.10262E-004
	NVSBBDF	0	73	73	6.47609E-007
	MVSBBDF	0	84	84	1.51782E-007
10^{-6}	NBDF	19	167	186	1.04720E-005
	VSBBDF	1	117	118	2.31132E-006
	NVSBBDF	0	142	142	6.61553E-009
	MVSBBDF	0	177	177	1.70621E-009

Table 4. Numerical Result for Problem 2

TOL	METHOD	TRS	TAS	TS	MAXE
10^{-2}	NBDF	4	65	69	9.38499E-002
	VSBBDF	0	45	45	6.48394E-004
	NVSBBDF	1	48	49	3.53673E-004
	MVSBBDF	0	47	47	3.14542E-004
10^{-4}	NBDF	7	107	114	1.02574E-004
	VSBBDF	0	63	63	6.48780E-005
	NVSBBDF	4	70	74	1.59269E-005
	MVSBBDF	0	72	72	1.59364E-005
10^{-6}	NBDF	10	134	144	1.03536E-005
	VSBBDF	0	99	99	6.48812E-006
	NVSBBDF	8	110	118	8.11971E-007
	MVSBBDF	0	127	127	7.22671E-007

Table 5. Numerical Result for Problem 3

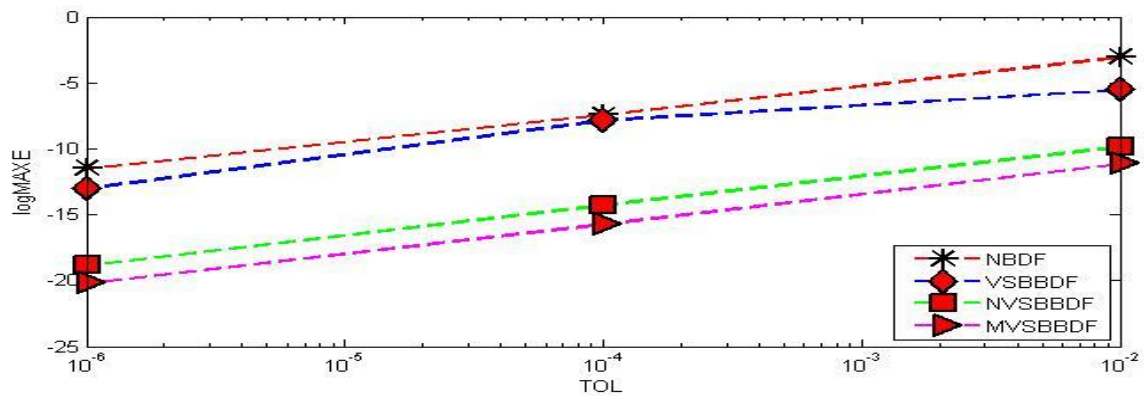
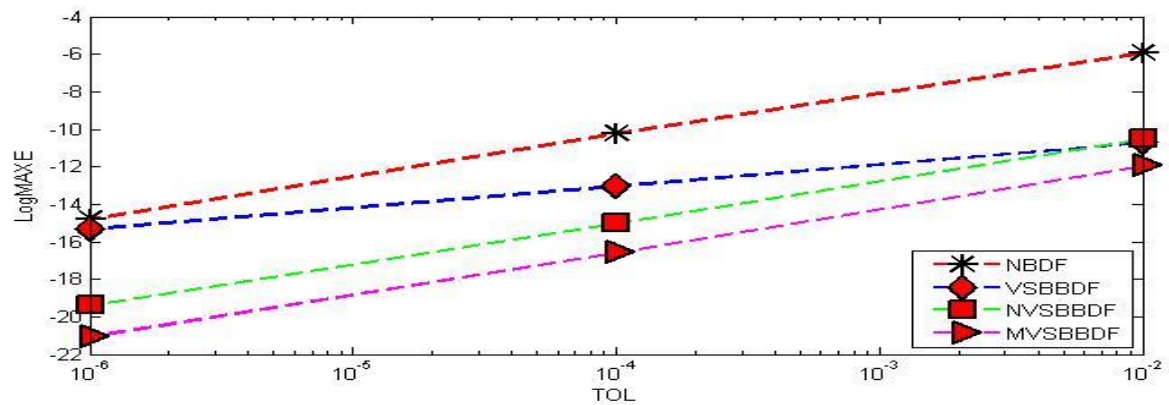
TOL	METHOD	TRS	TAS	TS	MAXE
10^{-2}	NBDF	3	187	190	1.04075E-002
	VSBBDF	1	79	80	2.58273E-002
	NVSBBDF	0	83	83	4.57917E-004
	MVSBBDF	0	92	92	3.48216E-004
10^{-4}	NBDF	3	255	258	2.39815E-004
	VSBBDF	5	141	146	1.61427E-004
	NVSBBDF	0	152	152	1.37209E-005
	MVSBBDF	0	198	198	2.20712E-006
10^{-6}	NBDF	4	390	394	3.28943E-006
	VSBBDF	9	180	189	2.01784E-006

NVSBBDF	3	187	190	2.45230E-007
MVSBBDF	6	189	195	2.31866E-007

Table 6: Numerical Result for Problem 4

TOL	METHOD	TRS	TAS	TS	MAXE
10^{-2}	NBDF	38	126	164	8.46687E-005
	VSBBDF	4	72	76	5.24918E-005
	NVSBBDF	1	65	66	5.16437E-005
	MVSBBDF	0	66	66	4.47900E-005
10^{-4}	NBDF	70	249	319	3.42565E-006
	VSBBDF	7	159	166	2.12255E-006
	NVSBBDF	0	116	116	1.54385E-006
	MVSBBDF	1	178	179	9.48730E-007
10^{-6}	NBDF	100	474	574	1.33439E-007
	VSBBDF	19	457	476	4.32393E-008
	NVSBBDF	4	398	402	3.11839E-008
	MVSBBDF	2	400	402	1.95286E-008

To visually demonstrate the performance of the MVSBBDF method, the accuracy graphs of $\text{Log}(\text{MAXE})$ against TOL are plotted in Figures 6-9.


Figure 6. Graph of $\text{Log}(\text{MAXE})$ against TOL for Problem 1

Figure 7. Graph of $\text{Log}(\text{MAXE})$ against TOL for Problem 2

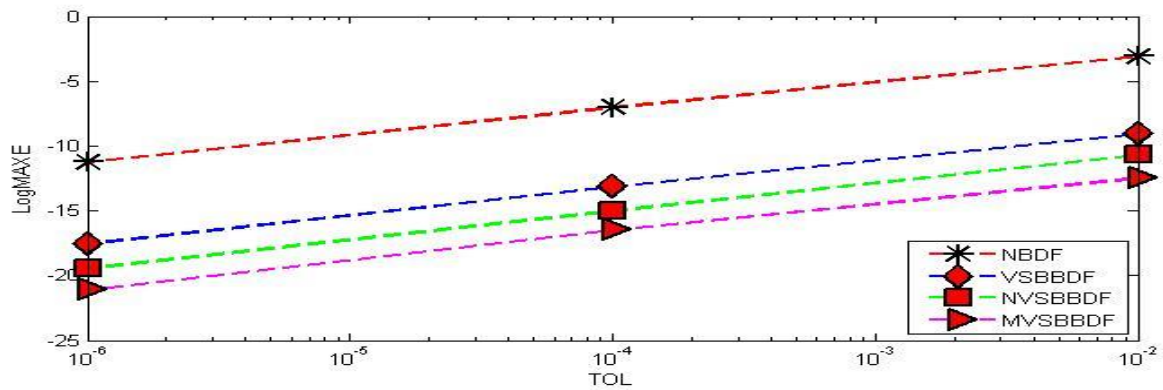


Figure 8. Graph of $\text{Log}(\text{MAXE})$ against TOL for Problem 3

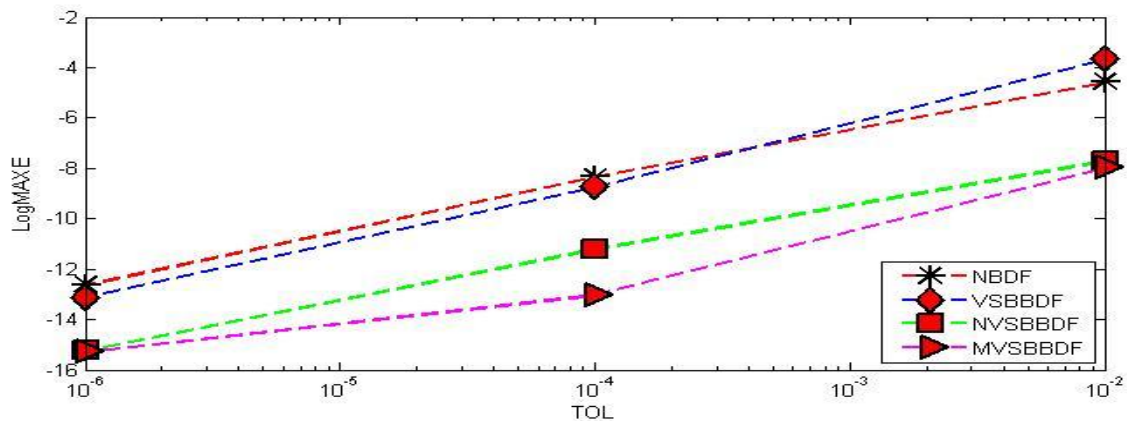


Figure 9. Graph of $\text{Log}(\text{MAXE})$ against TOL for Problem 4

4. Discussion on the Results

The numerical results presented in Tables 3-6 and illustrated in Figures 6-9 demonstrate the accuracy, efficiency, and stability of the MVSBBDF method when applied to a range of stiff and oscillatory problems. Across all test problems and tolerance levels, the method consistently produced very low maximum absolute errors (MAXE), which decreased proportionally as the tolerance (TOL) was reduced, confirming the strong convergence property of the proposed scheme. This consistent reduction in error highlights the method's ability to maintain stable and predictable error behavior even for highly stiff problems. Moreover, the MVSBBDF method exhibited an excellent stability profile, as reflected by the zero or near-zero failed steps (TRS) recorded across all problems and tolerance levels, indicating robust control of the adaptive step-size mechanism. Regarding computational efficiency, the total number of integration steps (TS) required by the MVSBBDF method increases moderately as the tolerance is tightened, reflecting the natural trade-off between accuracy and computational effort. This increase is primarily due to the positive term $(+\rho f_{n+k-1})$ introduced in the MVSBBDF modification, which replaces the corresponding term in the existing NVSBBDF formulation. While this adjustment slightly increases the number of steps, it significantly enhances both stability and accuracy. The adaptive step-size mechanism further optimizes performance by dynamically adjusting the step length based on local error estimates, ensuring precise and stable integration with minimal step rejections. The graphs behavior observed in the accuracy graphs of $\text{Log}(\text{MAXE})$ versus TOL plots (Figures 6-9) confirm the method's convergence properties and demonstrates that the MVSBBDF method reliably delivers accurate, stable, and efficient solutions across a wide range of stiff and oscillatory IVPs.

5. Conclusion

In this study, the MVSBBDF was developed for the numerical solution of first-order stiff and oscillatory IVPs. The method extends the two-point variable step-size superclass of block BDF by introducing a positive corrective term, which enhances stability and accuracy while maintaining computational efficiency. Theoretical analysis confirmed that the MVSBBDF method is fourth-order accurate, consistent, zero-stable, A-stable, and therefore convergent. The adaptive step-size mechanism, combined with a predictor-corrector implementation, allows the method to adjust dynamically to variations in stiffness and solution behavior, reducing step rejections and ensuring accurate approximations. Numerical experiments on a range of stiff and oscillatory test problems demonstrated that the MVSBBDF method achieves superior accuracy compared to existing block BDF methods, with minimal additional computational cost.

Overall, the MVSBBDF method provides a robust, accurate, and efficient framework for solving stiff systems of ordinary differential equations, making it a valuable tool for both theoretical and applied computations in scientific and

engineering applications. Future work may explore higher-order extensions and the application of the method to large-scale stiff systems.

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Declaration of Competing Interest

The authors declare no conflict of interest to report regarding this study conducted.

CRedit Authorship Contribution Statement

Alhassan Buhari (Conceptualisation; Methodology; Software; Validation; Formal analysis; Investigation; Resources; Data maintenance; Writing - creating the initial design; Writing - reviewing and editing; Visualisation and Project management)

Musa Hamisu (Conceptualisation; Methodology; Software; Validation; Formal analysis; Investigation; Resources; Data Maintenance; Writing - Creating the initial design; Writing - reviewing and editing; Visualisation; Monitoring and Project management)

Bala Najamuddeen (Formal analysis; Investigation and Writing-reviewing and editing)

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Availability of the Data and Materials

The data used to support the findings of this study are included within the article.

Ethical Declarations

No artificial intelligence tools were used in the preparation of this manuscript. All content was developed manually by the authors. This study did not involve human participants or animals. Ethical approval was therefore not required.

Generative Artificial Intelligence Declarations

The authors claim that artificially intelligent-assisted technologies, such as generative AI, were not used to generate content, ideas, or theories. We have just utilised AI to enhance readability and refine the language. This was used with extreme human control and oversight. The authors take full responsibility for reviewing and approving the content.

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