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RESEARCH ARTICLE

Modified magnetic flow of Williamson nanofluid over a stretching sheet with Newtonian heating

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ABSTRACT - In systems where heat transfer and flow dynamics are crucial, the Williamson nanofluid model is a useful tool for studying non-Newtonian fluids. Numerous industrial applications, such as energy systems, manufacturing processes, and thermal management systems, depend on non-Newtonian fluid flows. In conjunction to this, this paper aims to investigate numerically the boundary layer flow of a Williamson nanofluid across a stretching sheet under modified magnetic influence and Newtonian heating (NH). Thus, the goal of the study is to investigate how the velocity and temperature profiles are affected by several characteristics, such as the concentration of nanoparticles, the strength of the magnetic field, the Williamson fluid properties, and thermal boundary conditions. To solve the nonlinear ordinary differential equations resulting from the application of similarity variables to the governing partial differential equations, the Runge–Kutta–Fehlberg (RKF45) method is utilized. To comprehend their impact on the fluid's behaviour and heat distribution, the analysis assesses several pertinent physical characteristics. The findings show that while magnetic effect reduces velocity, raising the Williamson parameter improves thermal performance. It is anticipated that these discoveries will advance our knowledge of non-Newtonian nanofluid flows and offer guidance for improving engineering systems that employ them.

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1. INTRODUCTION

There are numerous engineering and industrial applications for studying non-Newtonian fluids. Non-Newtonian fluids are more complex and widely used in industries than Newtonian fluids, prompting numerous studies on them. These complex fluids include Carreau, Casson, Jeffrey, Maxwell, Oldroyd-B, power-law, third grade, Casson fluid, Williamson fluid, Nanofluid and to name a few. Its viscosity can change in response to applied forces, displaying phenomena like shear-thinning, shear-thickening, and viscoelasticity. Meanwhile, a Newtonian fluid's viscosity remains constant regardless of the amount of shear applied at a constant temperature. These fluids exhibit a linear relationship between their viscosity and shear stress.

One of the non-Newtonian fluids is Nanofluid, in which colloidal suspensions of base fluid and nanoparticles range from 1 to 100 nm. Nowadays, these fluids are the focus of research because these small particles can enhance the coefficient of heat transfer several times when compared to the base fluid [1]. These nanofluids have applications in a variety of fields, including biomedical equipment and industries, electronic, automobile, and fuel industries. The study of nanofluids provides chances to investigate novel fluid flow phenomena and improve heat transfer procedures. Research on using nanoparticles to improve fluids' thermal conductivity was presented by Choi and Eastman [2]. Many authors have also proposed models and approaches for analysing convective fluxes of nanofluid, including the most popular models put forth by Buongiorno [3] and Tiwari and Das [4]. The study of Buongiorno [3] said that nanofluids are designed colloids combining a base fluid and nanoparticles(1-100nm). Then, Tiwari and Das [4] found that the single-phase model, where both the fluid phase and the solid particles are in thermal equilibrium and move at the same local velocity. The dynamics of nanofluids are constantly investigated using the two mentioned quantitative models. Later, this approach was widely used for various nanofluid challenges involving expanding and contracting surfaces. For instance, Zaimi et al. [5] conducted a study on thermal transport flow of a nanofluid across a deformable surface. Recent advancements in the study of nanofluid flow have been conducted under certain conditions [6-9].

Magnetohydrodynamics (MHD) is a branch of fluid dynamics that examines the movement of an electrically conducting fluid in a magnetic field. Currently, magnetohydrodynamics is applied in astrophysics and geophysics, fission

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and fusion, metallurgy, and direct energy conversion. It has been widely employed in cooling operations, metalworking processes, traveling wave tubes, propulsion units, and other applications. The previous work related to MHD effect on Casson nanofluid is conducted by Reddy and Maddileti [10] where they investigated how the joule parameter affected the fluid flow across extending surface. Meanwhile, for the geometry of stretching cylinder with the effect of thermal radiation, magnetic field, and electromagnetic loss on Williamson fluid is conducted by Hayat et al. [11]. Using the same geometry, the numerical study on Williamson fluid flow with varying thermal conductivity is examined by Malik et al. [12]. On the other hand, an analytical study on the impact of the non-uniform magnetic field is performed by Hayat et al. [13], who have reviewed 3rd -grade Nano liquid stagnation point flow using the Homotopy Analysis Method (HAM). A recent study by Vinodkumar Reddy et al. [14] analysed the existence of a chemical reaction and the injection MHD radiative flow of a Williamson nanofluid using the Cattaneo-Christov model across a stretching sheet via a porous media.

According to the literature, nanofluid flow can occur in a wide range of geometries in various circumstances, including effects and base fluid. However, no explicit research has been undertaken on the modified MHD of Williamson nanofluid over a stretching sheet with NH as the boundary conditions in which previous works are solely focused on the transverse magnetic flow. Therefore, this study proposes a mathematical model for Williamson nanofluid across a stretching sheet using Newtonian heating with the application of acute angle to the magnetic field flow. This mathematical formulation builds upon the work of Tiwari and Das [4] and Arifin et al. [15]. The model incorporates Tiwari-Das interactions between non-Newtonian Williamson fluid and nanoparticles. To convert partial differential equations (PDEs) to ordinary differential equations (ODEs), a suitable similarity transformation is employed. Next, The ODEs will be numerically solved in MAPLE software using RKF45 method.

2. PROBLEM FORMULATION

The Williamson Nanofluid flows on stretching sheet in the presence of a modified MHD effect is taking into account in the current study. As $u = u_w(x) = ax$, a uniform stress causes the stretching sheet's flow. The constant α is assumed to be positive. The flow region of vertical stretching sheet, when an acute angle α_1 is applied along with the magnetic field, is shown in Figure 1. The thermal boundary condition of NH is introduced on the stretching surface.

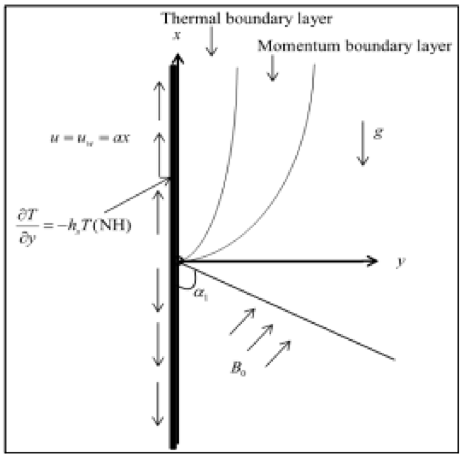

Figure 1. Physical configuration of Williamson nanofluid

The governing equations of Williamson nanofluid can be expressed as ([4], [15])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 u}{\partial y^2} + \sqrt{2}\frac{\mu_{nf}}{\rho_{nf}}\Gamma\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} - \frac{\sigma_{nf}}{\rho_{nf}}B^2 u\, sin^2\,\alpha_1 \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{nf}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

The following boundary conditions apply to the current fluid analysis:

$$\begin{gathered} = u_w(x) = ax, v = 0, \frac{\partial T}{\partial y} = h_s T\ at\ y = 0 \\ u \to 0, T \to T_\infty\ at\ y \to \infty \end{gathered} \tag{4}$$

where (u, v) are the velocity components of the fluid along x and y axes, respectively. T is the fluid temperature, ν is the kinematic viscosity, $\Gamma > 0$ is the time constant, ρ is the fluid density, α_1 is the aligned angle and B_0 is the magnetic-field strength. Table 1 displays the thermophysical properties of the nanofluid.

Table 1. Thermophysical properties of nanofluid [9]

Properties	Nanofluid
Dynamic viscosity	$\mu_{nf}=\frac{\mu_f}{(1-\varphi)^{2.5}}$
Density	$\rho_{nf} = (1-\varphi)\rho_f + \rho_{s,}$
Thermal diffusivity	$\alpha_{nf} = \frac{k_{nf}}{(\rho c_{p})_{nf}}$
Thermal conductivity	$\frac{k_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2\varphi(k_f - k_s)}{(k_s + 2k_f) + \varphi(k_f - k_s)}$
Heat capacity	$(\rho c_{p})_{nf} = (1-\varphi)(\rho c_{p})_f + \varphi(\rho c_{p})_s$

To simplify (1) -(4), the following similarity transformation are adopted

$$\eta = \left(\frac{a}{v_f}\right)^{\frac{1}{2}} y,\ \ \varphi=(av_f)^{\frac{1}{2}} x f(\eta),\ \ \theta(\eta) = \frac{T-T_\infty}{T_\infty}(NH) \tag{5}$$

where $\eta, \varphi\ and\ \theta$ are the non-dimensional similarity variables, stream function and temperature. Note that, by definition $u = \partial\varphi/\partial y$ and $v = \partial\varphi/\partial x$. The resulting equations can be expressed as

$$\frac{1}{(1-\phi)^{2.5}(1-\phi) + \phi\left(\frac{\rho_s}{\rho_f}\right)} f'''(\eta) + f(\eta)f''(\eta) + x_3 f''(\eta)f'''(\eta) - (f'(\eta))^2) - \mathrm{M}\, sin^2\,\alpha f'(\eta) = 0 \tag{6}$$

$$\frac{1}{p_r}\frac{k_{nf}/k_f}{(\rho C_p)_{nf}/(\rho C_p)_f}\ \theta''(\eta) + f(\eta)\theta'(\eta) = 0 \tag{7}$$

The transformed boundary conditions are

$$\begin{gathered} f(0) = 0, f'(0) = 1, \theta'(0) = \gamma\big(1 + \theta(0)\big) at\ \eta = 0 \\ f'(\eta) \to 0,\ \ \theta(\eta) \to 0\ \ as\ \eta \to \infty \end{gathered} \tag{8}$$

3. NUMERICAL PROCEDURE

The model is solved numerically after the mathematical formulation has been analysed. In conjunction to this, the governing equations (1) – (4) are reduced to ordinary differential equations (6) – (8) through the application of similarity transformation display by Eq. (5). Then, RKF45 method was used to solve Eqs. (6) – (8) in the MAPLE software due to its approach's ability to solve initial value problems (IVPs) with great accuracy. It is important to state that, this method is efficient and compatible to this type of problem as approved by Arifin et al. [15] and Zokri et al. [6], whose results fully satisfied the boundary conditions. Next, the model under consideration needs to be verified by contrasting it with the previous literature review. The numerical outcomes under consideration, which include the modified angle, magnetic field, types of nanoparticles on the temperature and velocity profiles are generated.

4. RESULTS AND DISCUSSION

To obtain the numerical results of the velocity profiles $(f'(\eta))$ and temperature profile $(\theta(\eta))$ for the Williamson nanofluid, respectively, a number of fixed relevant parameters must be used which are $\lambda = 1, Pr = 6.2, \alpha = \frac{\pi}{6}, M = 0.5, b = 0.3, \phi = 0.01$. The values used in this study are consistent with those in previous research and align with the proposed model, as evidenced by the asymptotic behavior of the generated graph. Table 2 shows the comparative results of skin friction coefficient for several values of M when $\phi = 0,\ \lambda = 0$ and $\alpha_1 = \frac{\pi}{2}$. To validate the numerical solutions found in this study, $-f''(0)$ has been computed for a range of M values. Therefore, a direct comparison between the current results and previous study published by Fathizadeh et al. [16], who utilized the RKF45 method to solve Casson fluid flow over a porous linearly stretching sheet, are conducted. From the table, there is a very good agreement, which proposes that the current model and its conclusions are acceptable.

The various values of Williamson parameter λ in respect to the velocity and temperature profiles were displayed in Figures 2 and 3. As seen in Figure 2, the yield stress in the velocity profile drops as the parameter λ value rises, causing the nanofluid to act more like a solid body. This finding contrasts with the temperature profile, which shows a rising tendency.

Table 2. Comparative values of skin friction coefficient $-f''(0)$ for various values of M when $\phi = 0,\ \lambda = 0,$ and $\alpha_1 = \frac{\pi}{2}$.

M	**Fathizadeh et al. [16]**	**Present**
0	1.00000	1.00048
1	1.41421	1.41422
5	2.44949	2.44949
10	3.31662	3.31662
50	7.14142	7.14142
100	10.0499	10.0499

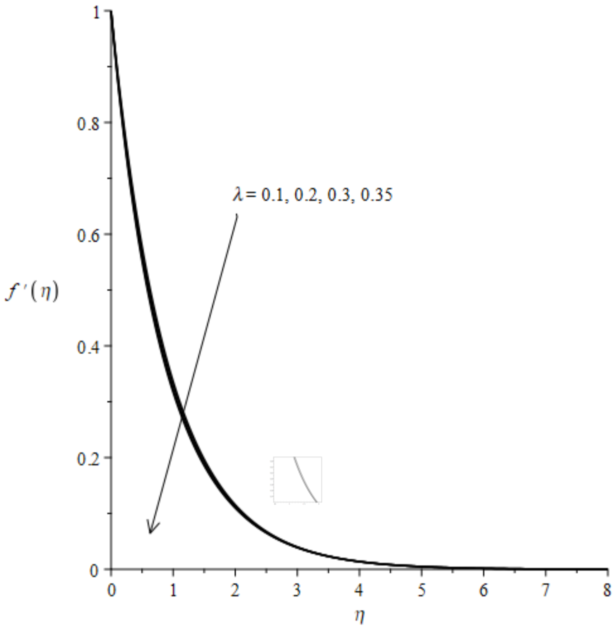

Figure 2. Velocity profile for various values of λ when $Pr = 6.2, \alpha = \frac{\pi}{6},\ M = 0.5, b = 0.3, \phi = 0.01$.

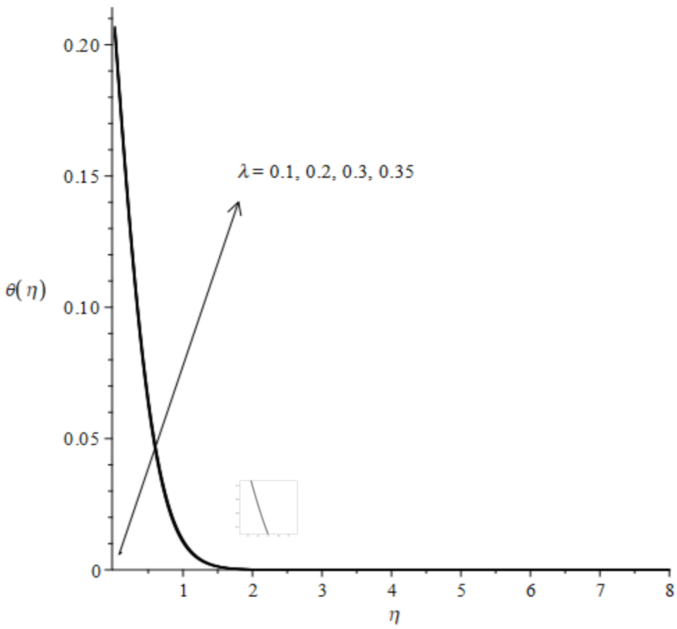

Figure 3. Temperature profile for various values of λ when $Pr = 6.2, \alpha = \frac{\pi}{6},\ M = 0.5, b = 0.3, \phi = 0.01$.

Figures 4 - 7 illustrate the velocity and temperature profiles of the Williamson nanofluid in relation to the effects of magnetic field, M, and aligned angle,a_1. From the Figures 4 and 7, as both parameters grow, the velocity profiles of fluid decrease. Noted that, Eq. (6) can be used to mathematically explain the correlation between the two parameters, M and a_1. The increasing of M is primely driven by the increment of a_1 and it led to assistance (Lorentz) force that moves opposed to the direction of the flow and reduces their movement in the flow of nanofluid. As shown in Figures 4 and .7, as the various values of the parameters M and a_1, the temperature profile for fluid seem to increase correspondingly.

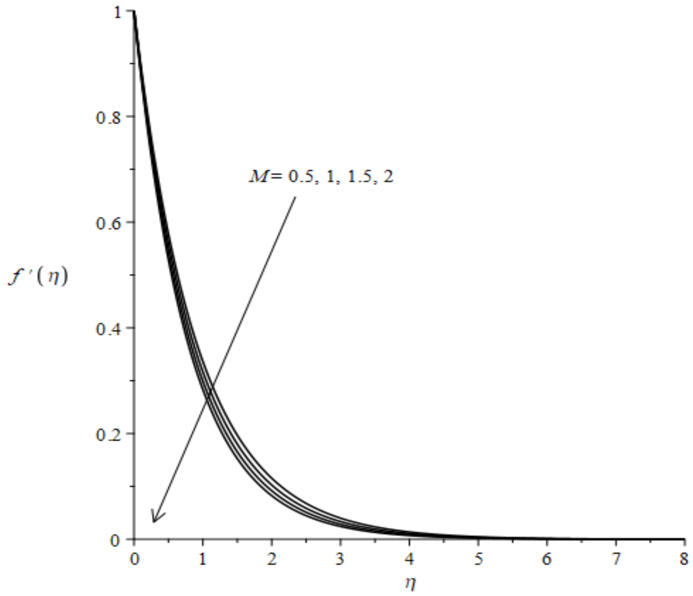

Figure 4. Velocity profile for various values of M when $Pr = 6.2, \alpha = \frac{\pi}{6}, \lambda = 1, b = 0.3$ and , $\phi = 0.01$.

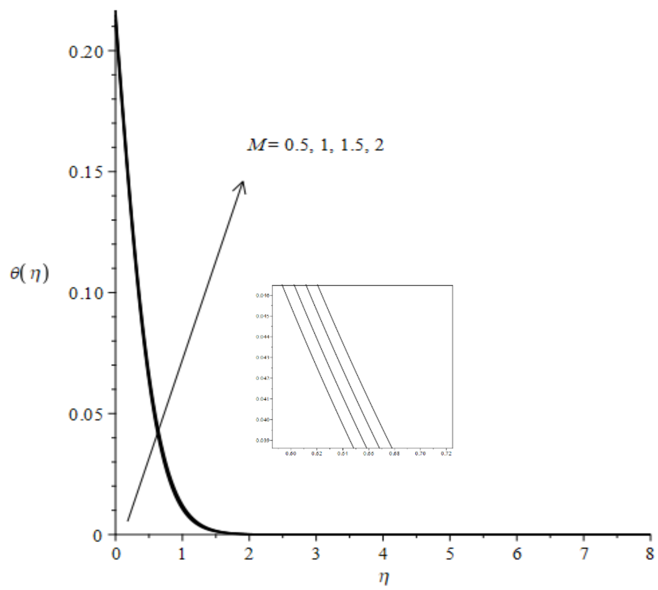

Figure 5. Temperature profile for various values of M when $Pr = 6.2, \alpha = \frac{\pi}{6}, \lambda = 1, b = 0.3,$ and $\phi = 0.01$.

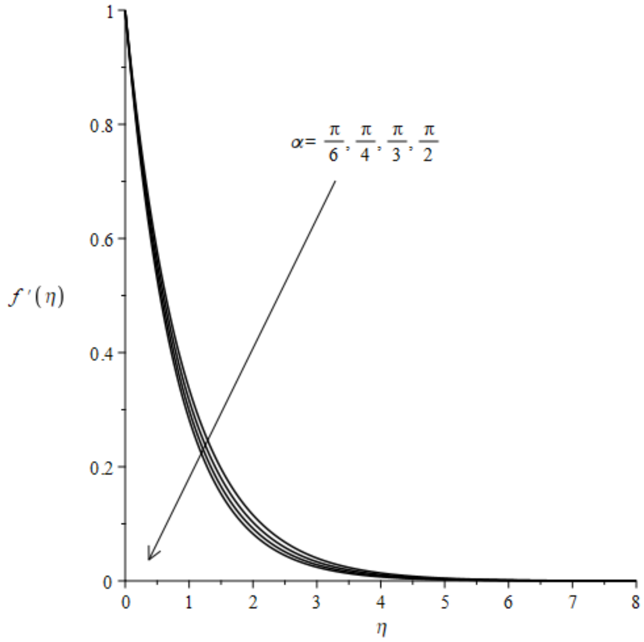

Figure 6. Velocity profile for various values of α when $Pr = 6.2, M = 0.5, \lambda = 1, b = 0.3$ and $\phi = 0.01$.

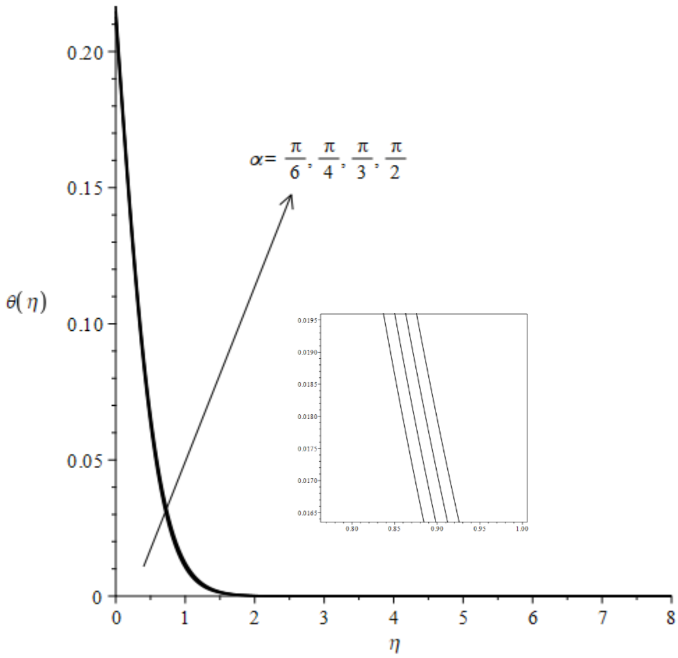

Figure 7. Temperature profile for various values of α when $Pr = 6.2, M = 0.5, \lambda = 1, b = 0.3, \phi = 0.01$ and $\phi = 0.01$.

Figures 8 and 9 portray the variation value of the volume friction of nanoparticles, ϕ on velocity and temperature profiles. Figure 8 show that, as ϕ increases, the velocity profile decreases. Meanwhile, a reverse trend can be observed for temperature profile. This phenomenon can be physically explained as an increase in thermal conductivity, which leads to the thickening of the thermal boundary layer.

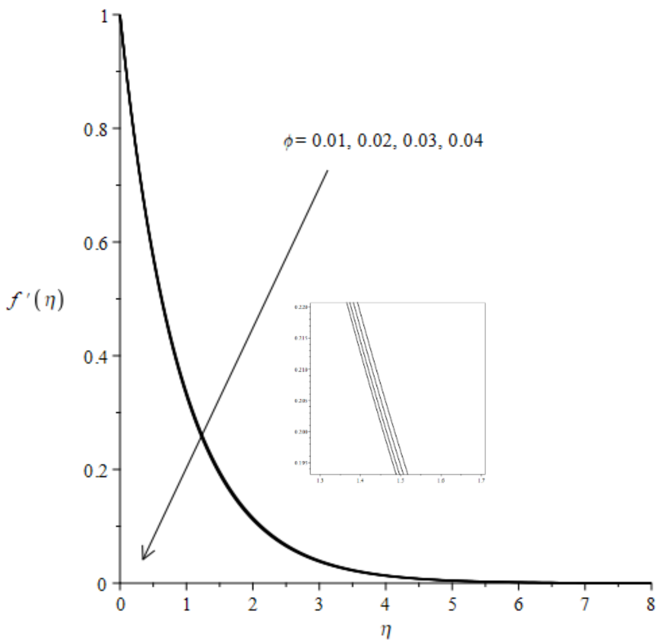

Figure 8. Velocity profile for various values of ϕ when $Pr = 6.2, M = 0.5, \lambda = 1, b = 0.3$ and $\alpha = \frac{\pi}{6}$.

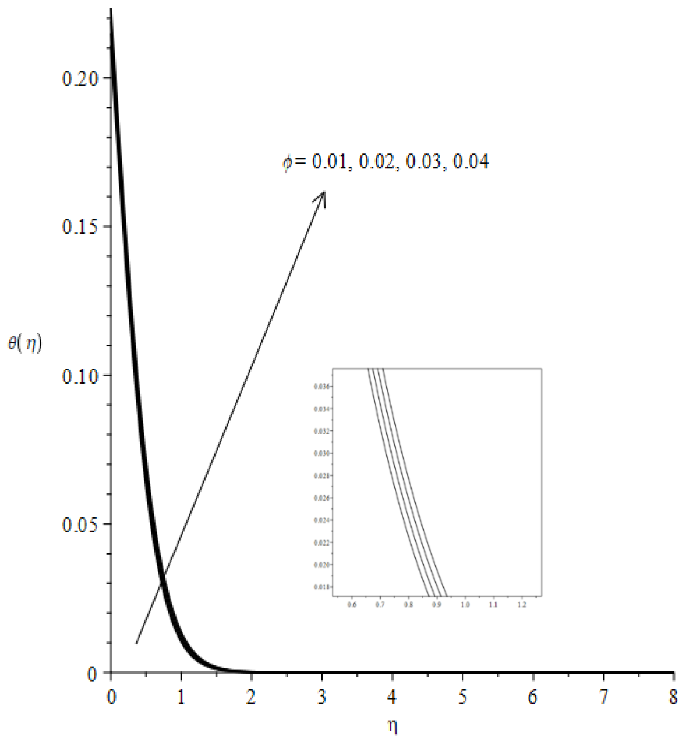

Figure 9. Temperature profile for various values of ϕ when $Pr = 6.2, M = 0.5, \lambda = 1, b = 0.3$ and $\alpha = \frac{\pi}{6}$.

4. CONCLUSIONS

The present work examines the modified MHD effect on Williamson fluid flows over a stretching surface with the presence of nanoparticles and NH. The results are obtained using the RKF45 numerical scheme. In conclusion, the results of the velocity and temperature profiles for several relevant parameters have been displayed in the form of graph coding generated by Maple software. These graphs illustrate the relationship between many different parameters. The main outcomes of this research can be summed up as follows**:**

- Increases in parameters ϕ, λ, α_1 and M reduce the velocity profile but increase the temperature profile.
- The temperature profile at the surface of the sheet for ϕ is higher compared to the other parameters, which confirms that the presence of nanoparticles enhances the thermal conductivity of the fluid. This finding is a key result of the current theoretical study and is in good agreement with previous experimental works.

Future research is recommended to investigate the behavior of nanoparticles in other types of non-Newtonian fluids to explore their physical properties under various conditions.

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AUTHOR CONTRIBUTIONS

Nur Syamilah Arifin (Derivation of mathematical model, Writing-original draft), Abdul Fatah Zainuddin (Problem Formulation), Ahmad Khudzairi Khalid (Numerical Computation), Nur Intan Syafinaz Ahmad (Numerical Analysis), Wan Munirah Wan Mohamad (Writing), Abdul Rahman Mohd Kasim (Supervision, Numerical Analysis), Imran Ullah (Supervision)

DECLARATION OF ORIGINALITY

The authors declare no conflict of interest to report regarding this study conducted.

GENERATIVE AI DECLARATIONS

The authors have not utilised any AI resources or tools in the preparation of this article.

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