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Generalized Γ-Cancellativity of Γ-AG-Groupoids

Thiti Gaketem
School of Science,University of Phayao, Phayao, 56000
E-mail address: newtonisaac41@yahoo.com

Abstract-In this paper we study some properties of a Γ–cancellativity on a Γ-AG-groupoid. Finally we study quasi-Γ-cancellativity which is a generalization of a Γ-cancellativity.

Keywords Γ-AG-groupoid, Γ-cancellativity, quasi- Γ-cancellativity

1. Introduction

Defintion 1.1 [1. P.41]. A groupoid $(S,\cdot)$ is called an *AG*-groupoid, if it satisfies left invertive law

$$(ab)c = (cb)a \qquad \text{for all } a,b,c \in S.$$

Lemma 1.2 [1. P.41]. An AG-groupoid S, is called a *medial law* if it satisfies

$$(ab)(cd) = (ac)(bd) \qquad \text{for all } a,b,c,d \in S.$$

Definition 1.3 [8. P.110]. An AG-groupoid S, is called a *paramedical* if it satisfies

$$(ab)(cd) = (db)(ca) \qquad \text{for all } a,b,c,d \in S.$$

Proposition 1.4 [2. P.110]. If S is an AG-groupoid with left identity, then

$$a(bc) = b(ac) \qquad \text{for all } a,b,c,d \in S.$$

Definition 1.5. [8, p.268] Let S and Γ be any non-empty sets. We call S to be Γ-AG-groupoid if there exists a mapping $S\times\Gamma\times S \to S$, written (a,α,b) by $a\alpha b$, such that S satisfies the identity $(a\alpha b)\beta c = (c\alpha b)\beta a$ for all $a,b,c \in S$ and $\alpha,\beta \in \Gamma$.

Definition 1.6. [4, p.2]. Let S and Γ be any non-empty sets. If there exists a mapping $S\times\Gamma\times S \to S$, written (a,α,b) by $a\alpha b$, S is called a Γ-*medial* if it satisfies $(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d)$ and called *a* Γ-*paramedial* if it satisfies $(a\alpha b)\beta(c\gamma d) = (d\alpha b)\beta(c\gamma d)$ for all $a,b,c,d \in S$ and $\alpha,\beta,\gamma \in \Gamma$.

Shan M. was introduced the concepts of cancellativity and quasi-cancellativity of an AG-groupoids as follows.

Definition 1.7. [7. P2188]. An element a of a AG-groupoid S is called *left cancellative* if $ax = ay$ implies that $x = y$ for all $x,y \in S$. Similarly an element a of a AG-groupoid S is called *right cancellative* if $xa = ya$ implies that $x = y$ for all $x,y \in S$. An element a of an AG-groupoid S is called *cancellative* if it is both left and right cancellative.

Definition 1.8 [6. P2066]**.** An AG-groupoid S is a *quasi-cancellative* if for any $x,y \in S$,

(1) $x = xy$ and $y^2 = yx$ implies that $x = y$,

(2) $x = yx$ and $y^2 = xy$ implies that $x = y$.

2. Γ-Cancellativity of Γ-AG-groupoids

In this paper, we introduce the concept of a Γ-cancellativity of Γ-AG-groupoids which is defined analogous to [6.] and investigate its properties.

Definition 2.1. An element a of a Γ-AG-groupoid S is called *left* Γ*-cancellative* if $a\alpha x = a\alpha y$ implies that $x = y$ for all $x, y \in S$ and $\alpha \in \Gamma$. Similarly an element a of a Γ-AG-groupoid S is called *right* Γ*-cancellative* if $x\alpha a = y\alpha a$ implies that $x = y$ for all $x, y \in S$ and $\alpha \in \Gamma$. An element a of a Γ-AG-groupoid S is called Γ*-cancellative* if it is both left and right Γ-cancellative.

Theorem 2.2. The following statements are equivalent for a Γ-AG-groupoid S:
(1) S is left Γ-cancellative;
(2) S is right Γ-cancellative;
(3) S is Γ-cancellative.

Proof. (1)$\Rightarrow$(2) Let S be left Γ-cancellative. Let a be an arbitrary element of S and let $x\alpha a = y\alpha a$. Let $k \in S$ and $\beta \in \Gamma$ for all $x, y \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}(k\alpha a)\beta x &= (x\alpha a)\beta k && \text{by Definition 1.1}\\ &= (y\alpha a)\beta k\\ &= (k\alpha a)\beta y && \text{by } \Gamma\text{-medial.}\end{aligned}$$

By left Γ-cancellativity, $x = y$. Thus S is right Γ-cancellative.

(2)$\Rightarrow$(3) Let S be right Γ-cancellative. Let a be an arbitrary element of S and let $a\alpha x = a\alpha y$. Let $k \in S$ and $\beta \in \Gamma$ for all $x, y \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}[(x\beta k)\alpha a]\gamma a &= (a\alpha a)\gamma(x\beta k) && \text{by Definition 1.1}\\ &= (a\alpha x)\gamma(a\beta k) && \text{by } \Gamma\text{-medial}\\ &= (a\alpha y)\gamma(a\beta k)\\ &= (a\alpha a)\gamma(y\beta k) && \text{by } \Gamma\text{-medial}\\ &= [(y\beta k)\alpha a]\gamma a && \text{by Definition 1.1.}\end{aligned}$$

By right Γ-cancellative, $x = y$. Thus S is left Γ-cancellative. Hence S is Γ-cancellative.

(3)$\Rightarrow$(1) This is clear. □

Theorem 2.3. Every right Γ-cancellative element of a Γ-AG-groupoid S is a left Γ-cancellative.

Proof. Let S be a Γ-AG-groupoid and let a be an arbitrary right Γ-cancellative element of S .Suppose that $a\alpha x = a\alpha y$ for all $a, x, y \in S$ and $\alpha \in \Gamma$. For $\beta, \gamma \in \Gamma$, we have

$$\begin{aligned}[(x\beta a)\alpha a]\gamma a &= (a\alpha a)\gamma(x\beta a) && \text{by Definition 1.1}\\ &= (a\alpha x)\gamma(a\beta a) && \text{by } \Gamma\text{-medial}\\ &= (a\alpha y)\gamma(a\beta a)\\ &= (a\alpha a)\gamma(y\beta a) && \text{by } \Gamma\text{-medial}\\ &= [(y\beta a)\alpha a]\gamma a && \text{by Definition 1.1.}\end{aligned}$$

Thus the right Γ-cancellativity implies that $x = y$. Hence a is left Γ-cancellative. Therefore every right Γ-cancellative element of S is left Γ-cancellative. □

Definition 2.4. [8. p269] An element $e \in S$ is called a *left identity* of a Γ-AG-groupoid if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.

The following two theorems are analogously to the in [7, p.2190].

Theorem 2.5. Let S be a Γ-AG-groupoid with a left identity e which is right Γ-cancellative. If $a\alpha b = c\alpha d$, then $b\gamma a = d\gamma c$ for all $a,b,c,d \in S$ and $\alpha,\gamma \in \Gamma$.

Proof. Let $a,b,c,d \in S$ and $\alpha,\gamma \in \Gamma$. Then by Definitions 1.1 and 2.4, we have the following implication

$$a\alpha b = c\alpha d \Rightarrow (e\gamma a)\alpha b = (e\gamma c)\alpha d \Rightarrow (b\gamma a)\alpha e = (d\gamma c)\alpha e .$$

Since S is right Γ-cancellative, thus $b\gamma a = d\gamma c$. We complete the proof. □

Theorem 2.6. Let S be a Γ-AG-groupoid with a left identity e. Then every left Γ-cancellative element is also right Γ-cancellative.

Proof. Let a be an arbitrary left Γ-cancellative element of S and suppose that $x\alpha a = y\alpha a$ for all $x, y \in S$ and $\alpha \in \Gamma$. Then by Theorem 2.5, we have $a\alpha x = a\alpha y$. Since S is left Γ-cancellative, $x = y$. Thus a is right Γ-cancellative. Hence every left Γ-cancellative element of S is right Γ-cancellative. □

A left invertible property of a Γ-AG-groupoid is defined analogously to AG-groupoid as in [5.p387].

Definition 2.7. Let S be a Γ-AG-groupoid with a left identity e. An element a of S is said to be *left invertible* if there exists an element a^{-1} of S such that $a^{-1}\alpha a = e$ for all $\alpha \in \Gamma$. In this case a^{-1} is called a *left inverse* of a. Dually, an element a of S is said to be *right invertible* if there exists an element a^{-1} of S such that $a\alpha a^{-1} = e$ for all $\alpha \in \Gamma$, a^{-1} is called a *right inverse* of a. If an element a of S is both left and right invertible, then a *is called invertible.*

Next, we prove that cancellativity and invertibility are coincident in a finite Γ-AG-groupoid S with a left identity e.

Theorem 2.8. Let S be a finite Γ-AG-groupoid S with a left identity e, then for all $a \in S$, a is invertible if and only if a is Γ-cancellative.

Proof. $(\Rightarrow)$ Assume that a is invertible. Then there exists $a^{-1} \in S$ such that $a^{-1}\alpha a = e = a\alpha a^{-1}$. Suppose that $x\alpha a = y\alpha a$ for all $x, y \in S$ and $\gamma \in \Gamma$. Then

$$x = e\gamma x = (a^{-1}\alpha a)\gamma x = (x\alpha a)\gamma a^{-1} = (y\alpha a)\gamma a^{-1} = (a^{-1}\alpha a)\gamma y = e\gamma y = y$$

Thus a is right Γ-cancellative. By Theorem 2.2, a is Γ-cancellative.

$(\Leftarrow)$ Assume that a is Γ-cancellative and let $S = \{s_1, s_2, \ldots, s_n\}$. Then for all $\alpha \in \Gamma$, $a\alpha s_1, a\alpha s_2, \ldots, a\alpha s_n$ are al l distinct. Since S is finite, there must exists a positive integer $i \in \{1, 2, \ldots, n\}$ such that $a\alpha s_i = e$ but then $s_i\alpha a = e$. By Theorem 2.5, we have $a\alpha s_i = s_i\alpha a = e$ for all $\alpha \in \Gamma$. Hence a is invertible. □

3. The Quasi-Γ-Cancellativity of a Γ-AG-groupoid

In section, we study definition of a quasi-Γ-cancellativity which is defined analogously as in [6. P2066] and also investigate its properties.

Definition 3.1. A Γ-AG-groupoid S is a *quasi-Γ-cancellative* if for any $x, y \in S$ and $\gamma \in \Gamma$,

(1) $x = x\gamma y$ and $y = y\gamma x$ implies that $x = y$,
(2) $x = y\gamma x$ and $y = x\gamma y$ implies that $x = y$.

Definition 3.2. A Γ-AG-groupoid S is said to be a Γ-*idempotent.* If $x\gamma x = x$ for all $x \in S$ and $\gamma \in \Gamma$.

Definition 3.3. A Γ-AG-groupoid S is said to be a Γ-*AG-band* if every element of *S* is a Γ-idempotent.

The following two theorems are analogously as in [6, p.2067-2068].

Theorem 3.4. Every Γ-AG-band is a quasi-Γ-cancellative.

Proof. Let S be a Γ-AG-band and let $x, y \in S$ and $\gamma, \beta \in \Gamma$. We shall show that S is a quasi-Γ-cancellative consider the following:
(1) Assume that $x = x\beta y$ and $y = y\beta x$. Then $x = x\gamma y$ and $y = y\gamma x$. Now

$$\begin{aligned} x &= x\beta y \\ &= (x\gamma x)\beta y && \text{by } x\gamma x = x \\ &= (y\gamma x)\beta x && \text{by Definition 1.1} \\ &= y\beta x && \text{by Definition 3.2} \\ &= y && \text{by } y = y\beta x. \end{aligned}$$

(2) Assume that $x = y\beta x$ *and* $y = x\beta y$. Then $x = y\gamma x$ and $y = x\gamma y$. Now

$$\begin{aligned} x &= y\beta x \\ &= (y\gamma y)\beta(x\gamma x) && \text{by Definition 3.2} \\ &= (x\gamma y)\beta(x\gamma y) && \text{by } \Gamma\text{-paramedical} \\ &= (x\gamma x)\beta(y\gamma y) && \text{by } \Gamma\text{- medial} \\ &= x\beta y && \text{by Definition 3.2} \\ &= y && \text{by } y = x\beta y \end{aligned}$$ □

Theorem 3.5. Let S be a Γ-AG-groupoid such that S is quasi- Γ-cancellative and $x\gamma x = x$ for all $x \in S$ and $\gamma \in \Gamma$. If S is Γ-medial, then the following statements hold:
(1) $x\gamma a = x\gamma b$ if and only if $a\gamma x = b\gamma x$,
(2) $(x\gamma y)\beta a = (x\gamma y)\beta b$ implies that $a\beta(y\gamma x) = b\beta(y\gamma x)$, for all $x, y, a, b \in S$ and $\gamma, \beta \in \Gamma$.

Proof. (1) $(\Rightarrow)$ Let $x\gamma a = x\gamma b$. Then $(x\gamma a)\beta(x\gamma a) = (x\gamma b)\beta(x\gamma a)$ and $(x\gamma a)\beta(x\gamma a) = (x\gamma b)\beta(x\gamma b)$. So

$$\begin{aligned} a\beta x &= (a\gamma a)\beta x && \text{by Definition 3.2} \\ &= (x\gamma a)\beta a && \text{by Definition 1.1} \\ &= (x\gamma b)\beta a && \text{by } x\gamma a = x\gamma b \\ &= (a\gamma b)\beta x && \text{by Definition 1.1} \\ &= (a\gamma b)\beta(x\gamma x) && \text{by Definition 3.2} \\ &= (a\gamma x)\beta(b\gamma x) && \text{by } \Gamma\text{–medial.} \end{aligned}$$

And

$$\begin{aligned} b\beta x &= (b\gamma b)\beta x && \text{by Definition 3.2} \\ &= (x\gamma b)\beta b && \text{by Definition 1.1} \end{aligned}$$

$= (x\gamma a)\beta b$ by $x\gamma a = x\gamma b$

$= (b\gamma a)\beta x$ by Definition 1.1

$= (b\gamma a)\beta(x\gamma x)$ by Definition 3.2

$= (b\gamma x)\beta(a\gamma x)$ by Γ-medial.

Then $a\beta x = (a\gamma x)\beta(b\gamma x)$ and $b\beta x = (b\gamma x)\beta(a\gamma x)$. Thus $a\gamma x = b\gamma x$.

$(\Leftarrow)$ This can be proved similarly.

(2.) Let $x, y, a, b \in S$ and $\gamma, \beta \in \Gamma$ such that $(x\gamma y)\beta a = (x\gamma y)\beta b$. Then $a = b$. So we have

$$a\beta(x\gamma y) = (a\gamma a)\beta(x\gamma y) = (a\gamma b)\beta(x\gamma y)$$
$$(x\gamma y)\beta a = (x\gamma y)\beta(a\gamma a) = (x\gamma y)\beta(a\gamma b)$$
$$(y\gamma x)\beta a = (y\gamma x)\beta(a\gamma a) = (y\gamma x)\beta(a\gamma b)$$
$$\Rightarrow a\beta(y\gamma x) = [a\beta(y\gamma x)]\alpha[a\beta(y\gamma x)] = [b\beta(y\gamma x)]\alpha[a\beta(y\gamma x)] \qquad (3.1)$$

Similarly, if $(x\gamma y)\beta a = (x\gamma y)\beta b$, then $a = b$, and so

$$b\beta(x\gamma y) = (b\gamma b)\beta(x\gamma y) = (b\gamma a)\beta(x\gamma y)$$
$$(x\gamma y)\beta b = (x\gamma y)\beta(b\gamma b) = (x\gamma y)\beta(b\gamma a)$$
$$(y\gamma x)\beta b = (y\gamma x)\beta(b\gamma b) = (y\gamma x)\beta(b\gamma a)$$
$$(a\gamma b)\beta(y\gamma x) = (b\gamma b)\beta(y\gamma x) = b\beta(y\gamma x)$$
$$\Rightarrow b\beta(y\gamma x) = [b\beta(y\gamma x)]\alpha[b\beta(y\gamma x)] = [a\beta(y\gamma x)]\alpha[b\beta(y\gamma x)] \qquad (3.2)$$

From (3.1), and (3.2) we have $a\beta(y\gamma x) = b\beta(y\gamma x)$. □

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