DATA ANALYTICS AND APPLIED MATHEMATICS (DAAM)
e-ISSN: 2773-4854
VOL. 3, ISSUE 1, 042 – 051
DOI: https://doi.org/10.15282/daam.v3i1.7617

ORIGINAL ARTICLE

A differential game of pursuit-evasion with constrained players' energy

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ABSTRACT – In the Hilbert space ℓ_2, we investigate a pursuit-evasion differential game involving countable number of pursuers and one evader. Players move in agreement with certain n[th] order ordinary differential equations with control functions of players satisfying integral constraints. The period of the game, which is denoted as θ, is fixed. During the game, pursuers want to minimize the distance to the evader and the evader want to maximize it. The game's payoff is the distance between evader and closest pursuer at time θ. Independent of relationship between energy resources of the players, we provide formula that defines value of the game and constructed players' optimal strategies.

ARTICLE HISTORY
Received: 09/01/2022
Revised: 23/03/2022
Accepted: 31/03/2022

KEYWORDS
Pursuer
Evader
Game Value
Constrained Energy
Player's Strategy

INTRODUCTION

Since the birth of differential games as a research area, the study of pursuit-evasion games has been a great area of interest to many researchers. This resulted into tremendous contributions in the research area. For example, the works [1-19] and some references therein are all concerned with the study of pursuit-evasion game problems. In these works, players' equations of motion are given as ordinary or partial differential equations of certain order. The works in [20-24] and some references therein are concerned with game problems involving partial differential equations. In the other hand, problem[1-8] and [10-18] involve ordinary differential equations of various orders. In particular, players' motion in the works [1], [10-15], [20] and [25-26] obeyed certain first order differential equations. The game problems in [4], [6-8] and [26], players' equations of motion are given as second order and higher order differential equations respectively.

The main focus of this paper is on the works such as [1], [4-5] and [25-26] that are concerned with constructionn of players' strategies that are optimal and obtaining value of the game. Speciffically, the works in [4], [5], [25] and [27] involve with the study of pursuit-evasion game with many pursuers and one evader in some Hilbert spaces. Players' equwations of motion are given as first order equations of the type $z = w(t), z(0) = z_0$. In [4] the Hilbert space considered was R^n and the constraint on all players'control functions is geometric except some few but finite number of pursuers whose control functions are subject to integral constraints. The problems in [5], [25] and [27] are formulated in the space ℓ_2 with integral restriction on the control functions of the players considered in [25] and [27]. While geometric constraints are considered in [5]. In all the four papers, Optimal strategies of the players are constructed and game value was found.

The paper [6] is concerned with the study of a pursuit-evasion differential game problem of fixed duration with countably many pursuers and an evader in the Hilbert space ℓ_2. Equations of motion of the players obeyed second order differential equations of the type $\ddot{z} = w(t), z(0) = z_0$. Control functions of the players are subject to integral constraints. Suffcient condition for finding value of the game is obtained when pursuers use strategy of parallel approach. This result was improved in [8] by eliminating the condition under which the value of the game is obtained in the paper [6]. Moreover, the game problem studied in [6] but with geometric constraints on players' control functions is studied in [7] and obtained the game value.

In this paper, we consider a pursuit-evasion differential game problem in the Hilbert space ℓ_2. In the game problem, mobility of each player is described by a certain n^{th} order differential equation and control function subject to integral constraint. For applications sake, we allow the number n to take values from the set $\{1,2,3\}$. That is, the order of the differential equation to represent either speed, acceleration or jerk of the dynamical objects (players of the game). The n^{th} order equations generalize players' dynamic equations considered in some related works presented earlier in this section and some references therein.

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PROBLEM OF THE RESEARCH

In this part of the paper, we present the research problem and state the research question.

The space

We consider the infinite dimensional Hilbert space

$$\ell_2 = \left\{ a = (a_1, a_2, ...) : \sum_{i=1}^{\infty} a_i^2 < \infty \right\},$$

with inner product h $\langle .,. \rangle : \ell_2 \times \ell_2 \to R$ and norm $\|.\| : \ell_2 \to [0, +\infty)$ defined by

$$\langle a,b \rangle = \sum a_i b_i, \qquad \| a \| = \left(\sum_{i=1}^{\infty} a_i^2 \right)^{1/2}, a, b \in \ell_2$$

Players' equations of motion

Suppose that the pursuers $P_i, i \in I = \{1, 2, ..., m...\}$,

and the evader E move according to the following n^{th} order differential equations $(n \in \{1, 2, 3, \})$:

$$\begin{aligned} P_i &: \frac{d^n s_i}{dt^n} = u_i(t), s_i(0) = s_i^0, \frac{ds_i}{dt}(0) = s_i^1, ..., \frac{d^{n-1} s_i}{dt^{n-1}}(0) = s_i^{n-1}, \\ E &: \frac{d^n g}{dt^n} = v(t), g(0) = g^0, \frac{dg}{dt}(0) = g^1, ..., \frac{d^{n-1} g}{dt^{n-1}}(0) = g^{n-1}, \end{aligned} \tag{2.1}$$

where $s_i, s_i^0, ..., s_i^{n-1}, u_i, g, g^0, g_1, g^2, ..., g^{n-1}, v \in \ell_2, u_i = (u_{i1}, u_{i2}, ...)$ is a control parameter of the pursuer P_i and $v = (v_1, v_2, ...)$ is that of the evader E. The span of the game is represented by a fixed positive number θ.

Definitions

Definition 2.1. A permissible control of the i^{th} pursuer P_i is the function $u_i(\cdot), u_i : [0, \theta] \to l_2$, whose coordinates $u_{ik}[0, \theta] \to R, k = 1, 2, ...,$ are Borel measurable functions and

$$\int_0^{\theta} \|u_i(t)\|^2 dt \le p_i^2, \tag{2.2}$$

where p_i is the upper bound of the i^{th} pursuer's energy.

Definition 2.2. A permissible control of the evader E is the function $v(.), v : [0, \theta] \to \ell_2$, whose coordinates $v_k : [0, \theta] \to R, k = 1, 2, ...,$ are Borel measurable functions and

$$\int_0^{\theta} \|v(t)\|^2 dt \le q^2, \tag{2.3}$$

where q is the upper bound of the evader's energy.

The solutions to the players' equations of motions (2.1) depend on the chosen permissible controls $u_i(\cdot), i \in I$ and $v(\cdot)$ by the pursuers and evader respectively. These solutions are given by

$$s_i(t) = (s_{i1}(t), s_{i2}(t), ...), g(t) = (g_1(t), g_2(t), ...),$$

where the coordinates are given by

$$s_{ik}(t) = s_{ik}^0 + ts_{ik}^1 + \frac{t^2}{2!} s_{ik} + ... + \frac{t^{n-1}}{(n-1)!} s_{ik}^{n-1} + \int_0^t \int_0^{t_1} \int_0^{t_2} ... \int_0^{t_{n-1}} u_{ik}(s) ds dt_{n-1} ... dt_2 dt_1,$$

$$g_k(\theta) = g_k^0 + t g_k^1 + \frac{t^2}{2!} g_k^2 + \cdots + \frac{t^{n-1}}{(n-1)!} g_k^{n-1} + \int_0^t \int_0^{t_1} \int_0^{t_2} ... \int_0^{t_{n-1}} v_k(s) ds \, dt_{n-1} ... dt_2 \, dt_1.$$

It is verifiable that $s_i(\cdot), g(\cdot) \in C\ (0, \theta; l_2)$, where $C(0, \theta; l_2)$ is the space of functions $a(t) = (a_1(t), a_2(t), ...) \in l_2$, $t \ge 0$, such that $a(t)$ is a continuous function in the norm of the space l_2 and whose coordinates $a_k(t), k = 1, 2, ...$ are absolute continuous functions.

Definition 2.3. A strategy of the i^{th} pursuer P_i is a function $u_i^*(t, s_i, g, v)$, $u_i^* : [0, \infty) \times l_2 \times l_2 \times l_2 \to l_2$, such that the system

$$\frac{d^n s_i}{dt^n} = u_i^*(t, s_i, g, v), s_i(0) = s_i^0, \frac{ds_i}{dt}(0) = s_i^1, \dots, \frac{d^{n-1}s_i}{dt^{n-1}}(0) = s_i^{n-1},$$
$$\frac{d^n g}{dt^n} = v, g(0) = g^0, \frac{dg}{dt}(0) = g^1, \dots, \frac{d^{n-1}g}{dt^{n-1}}(0) = g^{n-1}$$

has a unique solution $(s_i(\cdot), g(\cdot))$, with $s_i(\cdot), g(\cdot) \in C(0, \theta, l_2)$, for arbitrary permissible control $v(\cdot)$, of the evader E. A strategy $u_i^*(\cdot)$ is permissible if each control involved in generating this strategy is permissible.

Definition 2.4. A strategy of the evader E is a function $v^*(t, s_1, s_2, \dots, g, v)$, $v^* : [0, +\infty) \times (l_2)^I \times l_2 \times l_2 \to l_2$, such that the system of equations

$$\frac{d^n s_i}{dt^n} = u_i(t), s_i(0) = s_i^0, \frac{ds_i}{dt}(0) = s_i^1, \dots, \frac{d^{n-1}s_i}{dt^{n-1}}(0) = s_i^{n-1}, i \in I,$$
$$\frac{d^n g}{dt^n} = v^*(t, s_1, \dots, s_r, \dots, g, v), g(0) = g^0, \frac{dg}{dt}(0) = g^1, \dots, \frac{d^{n-1}g}{dt^{n-1}}(0) = g^{n-1},$$

has a unique solution $(s_1(\cdot), s_2(\cdot), \dots, g(\cdot))$, with $s_i(\cdot), g(\cdot) \in C(0, \theta, l_2)$, for arbitrary permissible controls $u_i(\cdot)$ of the pursuers P_i. The strategy $v^*(\cdot)$ is permissible if each control involved in the formation of this strategy is permissible.

Definition 2.5. Optimal strategies of the pursuers $P_i, i \in I$, are strategies $\hat{u}_i^*, i \in I$ such that

$$\psi_1(\hat{u}_1^*, \hat{u}_2^*, \dots) = \inf_{u_1^*, u_2^*, \dots} \psi_1(u_1^*, u_2^*, \dots)$$

where $\psi_1(u_1^*, u_2^*, \dots) = \sup_{v(.)} \inf_{i \in I} \|s_i(\theta) - g(\theta)\|$; u_i^* are pursuers' permissible strategies and $v^*(\cdot)$ is the evader's permissible control.

Definition 2.6. An optimal control strategy of the evader E is the strategy $\hat{v}^*$ such that

$$\psi_2(\hat{v}^*) = \sup_{v^*(\cdot)} \psi_2(v^*),$$

where $\psi_2(v^*) = \inf_{u_1(.), u_2(.), \dots} \inf_{i \in I} \|s_i(\theta) - g(\theta)\|$; $u_i(\cdot)$, $i \in I$ are pursuers' permissible controls and v^* evader's permissible strategy. Subsequently, the game described by (2.1)-(2.3) will be called game $G(p_i, e)$. It is reported in [11] a number ς defines the value of the game $G(p_i, e)$, if

$$\Psi_1(\hat{u}_1^*, \hat{u}_2^*, \dots, \dots) = \varsigma = \Psi_2(\hat{v}^*).$$

Research problem: The problem is to construct formula for computing value of the game and establish optimal strategies of the players.

RESULTS

Results of this research are to be presented in this section. Prior to this, we present some existing results that are useful in the proof of the main result of the paper.

Preparatory results

Solutions of the players' equations of motion: The solution to the equation P_i in (2.1) is given by

$$s_i(\theta) = s_{io} + \int_0^\theta \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} u_i(t) dt dt_{n-1} \cdots dt_2 dt_1 \quad (3.1)$$

$$= s_{io} + \int_0^\theta \frac{(\theta - t)^{n-1}}{(n-1)!} u_i(t) dt, \quad (3.2)$$

where $s_{io} = s_i^0 + \theta s_i^1 + \frac{\theta^2}{2!} s_i^2 + \dots + \frac{\theta^{n-1}}{(n-1)!} s_i^{n-1}$ and the expression with the multiple integrals in (3.1) can be reduced to the expression with single integral in (3.2). The reduction method is discussed in the book [12]. The equation (3.2) is called the state equation of the i^{th} pursuer. In the same way, evader's state equation can also be obtained from (2.1) and is given by

$$g(\theta)=g_{_o}+\int_o^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}v(t)dt, \tag{3.3}$$

where $g_0=g^0+\theta g^1+\frac{\theta^2}{2!}g^2+\ldots+\frac{\theta^{n-1}}{(n-1)!}g^{n-1}$.

We should observe that the state equation (3.2) for the pursuer P_i and (3.3) of the evader E can be obtained from the following first order differential equations:

$$\begin{cases} P_i: \frac{ds_i}{dt}=\frac{(\theta-t)^{n-1}}{(n-1)!}u_i(t), s_i(0)=s_{i0}, \\ E: \frac{dg}{dt}=\frac{(\theta-t)^{n-1}}{(n-1)!}v(t), g(0)=g_0. \end{cases} \tag{3.4}$$

where

$$s_{i0}=s_i^0+\theta s_i^0+\frac{\theta^2}{2!}s_i^2+\ldots+\frac{\theta^{n-1}}{n-1}s_i^{n-1},$$

$$g_0=g^0+\theta g^1+\frac{\theta^2}{2!}g^2+\ldots+\frac{\theta^{n-1}}{n-1}g^{n-1}.$$

In view of this, in the game $G(\ p_i\ , e)$, players' equations of motion (2.1) can be replaced by (3.4).

Reachable sets

Players' reachable sets in the game G(, e) are given in the proposition below:

Proposition 3.1 *The reachable set of the*

1. i^{th} *pursuer* P_i *from the initial position* $s_{i0}\in\ell_2$ *at time* $t=0$ *is the ball* $B_{R_{P_i}}(s_{i0})$ *with center at* s_{i0} *and radius*

$R_{P_i}=\left(\frac{\theta^{2n-1}}{2n-1}\right)\frac{p_i}{(n-1)!}$. *That is*

$$B_{R_{P_i}}(s_{10})=\left\{s\in l_2:\left\|s-s_{i0}\right\|\le\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}\frac{p_i}{(n-1)!}\right\},$$

2. *evader E from the initial position* g_0 *at time t = 0 is the ball* $B_{R_{P_i}}(g_0)$ *with center at* g_0 *and radius*

$R_E=\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}\frac{p}{(n-1)!}$. That is

$$B_{R_E}(g_0)=\left\{g\in l_2:\left\|g-g_0\right\|\le\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}\frac{q}{(n-1)!}\right\}.$$

Proof To proof part 1, we show that $s_i(\theta)\in B_{R_{P_i}}(s_{i0})$ and for any $\tilde{s}\in B_{R_{P_i}}(s_{i0})$ there exist a control $u_i(t)$ such that $s_i(\theta)=\tilde{s}$. Firstly, using (3.2) and Cauchy-Schwarz inequality we have

$$\begin{aligned}\|s_i(\theta)-s_{i0}\|&=\left\|\int_0^\theta\frac{(\theta-t)^{n-1}}{(n-1)!}u_i(t)dt\right\| \\ &\le\frac{1}{(n-1)!}\int_0^\theta(\theta-t)^{n-1}\|u_i(t)\|dt \\ &\le\frac{1}{(n-1)!}\left(\int_0^\theta(\theta-t)\right)^{\frac{1}{2}}\left(\int_0^\theta\|u_i(t)\|^2\right)^{\frac{1}{2}} \\ &\le\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}\frac{p_i}{(n-1)!}.\end{aligned}$$

Secondly, using the fact that $\tilde{s}\in B_{R_{P_i}}(s_{i0})$ and letting

$$u_i(t)=\frac{(2n-1)}{\theta^{2n-1}}\frac{(n-1)!}{(\theta-t)^{1-n}}(\tilde{s}-s_{i0}), t\in[0,\theta],$$

we have

$$\begin{aligned} s_i(t) &= s_{i0} + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\frac{(2n-1)}{\theta^{2n-1}}\frac{(n-1)!}{(\theta-t)^{1-n}}(\tilde{s}-s_{i0})dt \\ &= s_{i0} + \left(\frac{2n-1}{\theta^{2n-1}}\right)\left(\tilde{s}-s_{i0}\right)\int_0^\theta (\theta-t)^{2n-2}dt \\ &= s_{i0} + \left(\frac{2n-1}{\theta^{2n-1}}\right)\frac{\theta^{2n-1}}{2n-1}\left(\tilde{s}-s_{i0}\right) \\ &= s_{i0} + \left(\tilde{s}-s_{i0}\right) = \tilde{s} \end{aligned}$$

This proves part 1 of the proposition. Using similar argument, we can prove part 2 of the proposition. With this, the proof of the proposition is complete.

Game of two players

We consider the game $G(p_i, e)$ with only one pursuer in place of countable number of pursuers. That is, a game problem with a pursuer and an evader with players' equations of motion given by

$$\begin{cases} P: \frac{ds}{dt} = \frac{(\theta-t)^{n-1}}{(n-1)!}u(t), s(0) = s^0 + \theta s^1 + \frac{\theta^2}{2!}s^2 + \ldots + \frac{\theta^{n-1}}{(n-2)!}s^{n-1} \\ E: \frac{dg}{dt} = \frac{(\theta-t)^{n-1}}{(n-1)!}v(t), g(0) = g_0 = g^0 + \theta g^1 + \frac{\theta^2}{2!}g^2 + \ldots + \frac{\theta^{n-1}}{(n-1)}g^{n-1}, \end{cases}$$

where $s_0 \neq g_0$ and the controls $u(t)$, $v(t)$ are such that

$$\int_0^\theta \|u(t)\|^2 dt \le p^2, \int_0^\theta \|v(t)\|^2 dt \le q^2.$$

The aim of the persuer P is to achieve the equation $s(\tau) = g(\tau)$ for some $\tau \in [0, \theta]$ and for the evader E is the opposite. Our goal is to find condition for which the pursuer can achieve its aim. To do this, we assume that $s_o \in \varphi$, where

$$\varphi = \left\{ z \in \ell_2 : 2\langle g_0 - s_0, z \rangle \le \left(\frac{\theta^{2n-1}(p^2-q^2)}{(2n-1)(n-1)!}\right) + \|g_0\|^2 - \|s_0\|^2 \right\},$$

Then, the following lemma gives sufficient condition that ensures the pursuer to achieve its aim:

Lemma 3.1 If $g(\theta) \in \phi$ then pursuer can achieve the equation $s(\theta) = g(\theta)$.
Proof The strategy of the persuer is defined by

$$u^*(t) = \frac{(2n-1)(g_0 - s_0)}{\theta^{2n-1}((n-1)!)^{-1}}(\theta-t)^{n-1} + v(t), t \in [0, \theta], \tag{3.5}$$

ensures the equation $s(\theta) = g(\theta)$. Indeed,

$$\begin{aligned} s(\theta) &= s_0 + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\left(\frac{(2n-1)(g_0-g_0)}{\theta^{2n-1}((n-1)!)^{-1}}(\theta-t)^{n-1} + v(t)\right)dt \\ &= s_0 + \frac{2n-1}{\theta^{2n-1}}(g_0 - s_0)\int_0^\theta (\theta-t)^{2n-2}dt + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}v(t)dt \\ &= s_0 + \frac{2n-1}{\theta^{2n-1}}(g_0 - s_0) + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}v(t)dt \\ &= s_0 + \frac{2n-1}{\theta^{2n-1}} + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}v(t)dt \\ &= g_0 + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}v(t)dt = g(\theta). \end{aligned}$$

We now establish establish an equality which is required in showing the permissibility of the strategy (3.5). Using the state equation of the evader (3.3) and the condition $g(\theta)\in\varphi$, we have

$$2\left\langle g_0 - s_0, \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}v(t)dt\right\rangle = 2\langle g_0 - s_0, g(\theta)-g_0\rangle$$

$$= 2\langle g_0 - s_0, g(\theta)\rangle - 2\|g_0\|^2 + 2\langle s_0, g_0\rangle$$

$$\le \left(\left(\frac{\theta^{2n-1}(p^2-q^2)}{2n-1((n-1)!)^2}\right) + \|g_0\|^2 - \|s_0\|^2\right) - 2\|g_0\|^2 + 2\langle s_0, g_0\rangle$$

$$\le \left(\frac{\theta^{2n-1}(p^2-q^2)}{2n-1((n-1)!)^2}\right) - \|g_0\|^2 - \|s_0\|^2 + 2\langle s_0, g_0\rangle$$

$$\le \left(\frac{\theta^{2n-1}(p^2-q^2)}{2n-1((n-1)!)^2}\right) - \|g_0 - s_0\|^2. \tag{3.6}$$

Using the inequality (3.6), we now show the permissibility of the pursuer's strategy (3.5) as follows:

$$\int_0^\theta \|u^*(t)\|^2 = \int_0^\theta \left\|\frac{(2n-1)(g_0-s_0)}{\theta^{2n-1}((n-1)!)^{-1}}(\theta-t)^{n-1} + v(t)\right\|^2 dt$$

$$= \int_0^\theta \left\|\frac{(2n-1)(g_0-s_0)}{\theta^{2n-1}((n-1)!)^{-1}}(\theta-1)^{n-1} + v(t)\right\|^2 dt$$

$$+\ 2\int_0^\theta \left\langle \frac{(2n-1)(g_0-s_0)}{\theta^{2n-1}((n-1)!)^{-1}}(\theta-t)^{n-1}, v(t)\right\rangle dt + \int_0^\theta \|v(t)\|^2 dt$$

$$\le \|g_0-s_0\|^2\left(\frac{2n-1}{\theta^{2n-1}}\right)^2((n-1)!)^2\int_0^\theta(\theta-1)^{2n-2}dt$$

$$+2\frac{(2n-1)((n-1)!)^2}{\theta^{2n-1}}\left\langle g_0 - s_0, \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}v(t)dt\right\rangle + q^2$$

$$\le \|g_0-s_0\|^2\left(\frac{2n-1}{\theta^{2n-1}}(n-1)!\right)^2\left(\frac{\theta^{2n-1}}{2n-1}\right)$$

$$+\frac{(2n-1)((n-1)!)^2}{\theta^{2n-1}}\left(\left(\frac{\theta^{2n-1}(p^2-q^2)}{2n-1((n-1)!)^2}\right) - \|g_0-s_0\|^2\right) + q^2$$

$$\le \|g_0-s_0\|^2\frac{2n-1((n-1)!)^2}{\theta^{2n-1}} + (p^2-q^2)$$

$$-\frac{2n-1((n-1)!)^2}{\theta^{2n-1}}\|g_0-s_0\|^2 + q^2 = p^2$$

This proves the Lemma.

Some important results

This part of the paper is meant for the presentation of some existing results that are useful in the proof of the main result of the paper.

Let the set $\varphi_i, i \in I_0 = \{i \in I : S_r(g_0) \cap B_{R_i}(S_{i0} \neq \emptyset\}$, be defined by

$$\varphi_i = \begin{cases} \{z \in l_2 : 2\langle g_0 - s_{i0}, z\rangle \le R_i^2 - r^2 + \|g_0\|^2 - \|s_{i0}\|^2\}, & \text{if } s_{i0} \neq g_0 \\ \{z \in l_2 : 2\langle z - g_0, z_0\rangle \le R_i\}, & \text{if } s_{i0} = g_0 \end{cases}$$

where $S_r(g_0) = \{g \in l_2 : \|g - g_0\| = r\}$ and $B_{R_i}(S_{i0}) = \{s \in l_2 : \|s - s_0\| \le R_i\}$. In view of this, we present the following lemmas:

Lemma 3.2. [18] Suppose there exist a non-zero vector $z_0 \in l_2$ such that $\langle g_0 - s_{i0}, z_0\rangle \ge 0$, for all $i \in I$. If $B_r(g_0) \subset \bigcup_{i\in I} B_{R_i}(s_{i0})$ then $H_r(g_0) \subset \bigcup_{i\in I_0} \varphi_i$.

Lemma 3.3. [14] Suppose there exists a non zero vector $z_0 \in l_2$, such that $\langle g_0 - s_{i0}, z_0\rangle \ge 0$, for all $i \in I$. Let $inf_{i\in I} R_i = R_0 > 0$ and for any $0 < \epsilon < R_0$ the set $\bigcup_{i\in I} B_{R_i-\epsilon}(s_{i0},)$ does not contain the ball $B_r(g_0)$ then there exists a point $\bar{g} \in S_r(g_0)$ such that $\|\bar{g} - s_{i0}\| \ge R_i$, for all $i \in I$.

Main results

In this subsection of the paper, we present the main result of the paper. That is, solution to the research problem which is given as theorem and its proof.

Theorem 3.1 If there exists a nonzero vector z_0 such that $\langle g_0 - s_{i0}, z_0\rangle \ge 0$, for all $i \in I$, then the number

$$\varsigma := \inf\{l \ge 0 : B_{R_E}(g_0)\} \subset \bigcup_{i\in I} B_{R_{P_i}} + l(s_{i0})\}, \tag{3.7}$$

where $R_E = \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{q}{(n-1)!}, R_{P_i} = \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{p_i}{(n-1)!}$, defines the value of the game $G(p_i, e)$.

Proof To proof this theorem we introduce fictitious pursuers with equations of motion given by

$$\frac{dz_i}{dt} = \frac{(\theta - t)^{n-1}}{(n-1)!}\omega_i(t), \ z_i(0) = s_{i0}, \qquad i \in I, \tag{3.8}$$

where the control functions $\omega_i(t), i \in I$ are such that

$$\left(\int_0^\infty \|\omega_i(t)\|^2\right)^{\frac{1}{2}} \le \bar{p}_i = p_i + \varsigma \left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}} (n-1)!. \tag{3.9}$$

The reachable set of the fictitious pursuer z_i from the initial state s_{i0} is the ball $B_r(s_{i0})$, with $r = \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{\bar{p}_i}{(n-1)!}$. Let strategies of the fictitious pursuers z_i, $i \in I$ be defined by

$$w_i^*(t) = \begin{cases} \frac{(2n-1)(g_0 - s_0)}{\theta^{2n-1}((n-1)!)^{-1}}(\theta - t)^{n-1} + v(t), & 0 \le t \le \tau, \\ 0, & \tau \le t \le \theta, \end{cases} \tag{3.10}$$

where τ is the time such that

$$\int_0^\tau \left\| \frac{(2n-1)(g_0 - s_0)}{\theta^{2n-1}\left((n-1)!\right)^{-1}}(\theta - t)^{n-1} + v(t) \right\|^2 dt = \bar{p}_i^{\,2}.$$

The strategy (3.10) will ensure the equation $z(\tau) = z_i(t) = z_i(\theta)$, $\tau \le t \le \theta$. Now we define the strategies of the real persuers by

$$u_i^*(t) = \frac{p_i}{\bar{p}_i}\omega_i(t), \ 0 \le t \le \theta, \tag{3.11}$$

The number ς defined by (3.7) is the value of the game $G(p_i, e)$, if the inequalities below hold

$$\sup_{v(.)} \inf_{i\in I} \|g(\theta) - s_i(\theta)\| \le \varsigma \le \inf_{u_1(.),u_2(.),\dots} \inf_{i\in I} \|s_i(\theta) - g(\theta)\|. \tag{3.12}$$

Now we proof the inequalities in (3.12). We begin with the proof of the left hand side inequality. By the definition of ς given in (3.7), we have

$$B_{R_E}(g_0) \subset \bigcup_{i\in I} B_{R_{P_i}+\varsigma}(s_{i0}).$$

Then by lemma 3.2 with $R_i = \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{p_i}{(n-1)!} + \varsigma$ and $r = \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{q}{(n-1)!}$, we have $B_{R_E}(y_0) \subset \bigcup_{i\in I_0} \varphi_i$,

where

$$\varphi_i = \begin{cases} \left\{z : 2\langle g_0 - s_{i0}, z\rangle \le \left(R_{P_i} + \varsigma\right)^2 - R_E^2 + \|g_0\|^2 - \|s_{i0}\|^2\right\}, & if\ s_{i0} \ne g_0 \\ \{z : \langle z - g_0, z_0\rangle \le R_{P_i} + \varsigma\},\ q \le \bar{p}_i, & if\ s_{i0} = g_0, \end{cases}$$

$$I_0 = \left\{ i \in I : S_{R_E}(g_0) \cap B_{R_{P_i}+\varsigma}(s_{i0}) \ne \emptyset \right\};\ S_{R_E}(y_0) = \{g \in l_2 : \|g - g_0\| = R_E$$

As a result of this, the evader will be in the set $\varphi_k, k \in I_0$, at time θ. That is $y(\theta) \in B_{R_E}(g_0) \subset \varphi_k, k \in I_0$.

This is for both the two definitions of the set φ_i. For the case $s_{k0} \ne g_0$ then we have in view of the Lemma 3.1, $z_k(\theta) = g(\theta)$ and the fictitious pursuer strategy (3.10) satisfies the inequality

$$\int_0^\infty \|\omega_k(t)\|^2\, dt \le \bar{p}_i^{\,2}.$$

In the other hand, when $s_{k0} = g_0$ and $\sigma \le \bar{p}_s$, we have the fictitious pursuer strategy (3.10) becomes $\omega_k^*(t) = v(t)$. This strategy will ensure the equation $z_k(\theta) = g(\theta)$. Moreover,

$$\int_0^\theta \|\omega_k(t)\|^2\, dt = \int_0^\theta \|v(t)\|^2\, dt \le q^2 \le \bar{p}_k^{\,2}.$$

Therefore, in both two cases, we have $z_k(\theta) = g(\theta)$. In view of this and the strategy of te real pursuer P_s defined by (3.11), we have

$$\begin{aligned}
\|s_k(\theta) - g(\theta)\| &= \|s_k(\theta) - z_k(\theta)\| \\
&= \left\| \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} u_k(t)dt - \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\, \omega_k(t)dt \right\| \\
&= \left\| \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} \frac{p_k}{\bar{p}_k}\, \omega_k(t)dt - \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\, \omega_k(t)dt \right\| \\
&= \left\| \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} \left(\frac{p_k}{\bar{p}_k} - 1\right) \omega_k(t)dt \right\| \\
&\le \left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}} \frac{\varsigma}{\bar{p}_k} \int_0^\theta (\theta - t)^{n-1} \|\omega_k(t)\|\, dt \\
&\le \left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}} \frac{\varsigma}{\bar{p}_k} \left(\int_0^\theta (\theta - t)^{2n-2}\, dt\right)^{\frac{1}{2}} \left(\int_0^\theta \|\omega_k(t)\|^2\, dt\right)^{\frac{1}{2}} \\
&\le \left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}} \frac{\varsigma}{\bar{p}_k} \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \bar{p}_k = \varsigma.
\end{aligned}$$

This proves the left hand inequality of (3.12) and it remains to show the right hand inequality. The right hand inequality is true if $\varsigma = 0$ for any permissible evader's strategy. We now consider the case when $\varsigma > 0$. In view of the definition of the number ς in (3.7), the set $\bigcup_{i\in I} B_{R_i^*}(s_{i0})$, where $R_i^* = R_{P_i} + \varsigma - \varepsilon$, for any $\varepsilon \in (0, \varsigma)$, does not contain the ball $B_{R_E}(g_0)$. Then by Lemma 3.2, there exists a point $\bar{g} \in S_{R_E}(g_0)$ $\left(i.e., \|\bar{g} - g_0\| = \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{q}{(n-1)!}\right)$ such that

$$\|\bar{g} - s_{i0}\| \ge \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{p_i}{(n-1)!} + \varsigma,\ i \in I. \tag{3.13}$$

Moreover, from the state equation of the pursuer (3.2), we have

$$\begin{aligned}\|s_i(\theta) - s_{i0}\| &= \left\| \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} u_i(t)dt \right\| \\ &\le \frac{1}{(n-1)!} \left(\int_0^\theta (\theta-t)^{2n-2} dt \right)^{\frac{1}{2}} \left(\int_0^\theta \|u_i(t)\| \, dt \right)^{\frac{1}{2}} \\ &\le \left(\frac{\theta^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \frac{p_i}{(n-1)!}, \quad i \in I. \end{aligned} \tag{3.14}$$

In view of (3.13) and (3.14), we have

$$\begin{aligned}\|\bar{g} - s_i(\theta)\| &\ge \|\bar{g} - s_{i0}\| - \|s_i(\theta) - s_{i0}\| \\ &\ge \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{p_i}{(n-1)!} + \varsigma - \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} \frac{p_i}{(n-1)!} = \varsigma,\end{aligned}$$

For all $i \in I$. In accordance with this, if $\bar{g} = g(\theta)$ then we have shown the right hand inequality in (3.12). The control

$$v(t) = q \left(\frac{2n-1}{\theta^{2n-1}} \right)^{\frac{1}{2}} (\theta - t)^{n-1} e, \qquad 0 \le t \le \theta,$$

where $e = \frac{\bar{g} - g_0}{\|\bar{g} - g_0\|}$, will bring the maximizing player to the point $\bar{g}$ at the time θ. This can be seen below

$$\begin{aligned} g(\theta) &= g_0 + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!} v(t)dt \\ &= g_0 + \frac{qe}{(n-1)!} \left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}} \int_0^\theta (\theta-t)^{2n-2} \, dt \\ &= g_0 + \frac{qe}{(n-1)!} \left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}} \frac{\theta^{2n-1}}{2n-1} \\ &= g_0 + \frac{qe}{(n-1)!} \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} = \bar{g}. \end{aligned}$$

This shows that the value of the game is indeed the number ς defined by (3.7). This is what brings us to the end of proof for the theorem.

CONCLUSION

The differential game studied consists of many pursuers and an evader in the space ℓ_2. Dynamic equation of each player is given as n^{th} order differential equation. Integral constrain imposed on control of each of the player translate to the energy or resource of each of the player been constrained. As we have seen, players' optimal strategies are constructed and we developed formula that defines the value of the game. The result of this paper is a generalization of some other results. For example, the results in [10] and [8] are corollaries to the result of this paper when n = 1 and n = 2 respectively.

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