DATA ANALYTICS AND APPLIED MATHEMATICS (DAAM)
e-ISSN:2773-4854
VOL.3, ISSUE 1,020 –027
DOI: https://doi.org/10.15282/daam.v3i1.7192

ORIGINAL ARTICLE

Stability analysis of mathematical model for the dynamics of diabetes mellitus and its complications in a population

P.O. Aye

Dept. of Mathematical Sciences, Adekunle Ajasin University, Akungba Akoko, Ondo State, Nigeria

ABSTRACT–This study present a mathematical model for the dynamics of diabetess and its complications in a population. The population under study was compartmentalized into healthy, susceptible, diabetic without complications, diabetic with complication and diabetic with complications undergoing treatment. The model is a system of linear ordinary differential equations. The stability of the model was investigated using Bellman and Coke method and the system was found to be stable.

ARTICLE HISTORY
Received:01/12/2021
Revised:28/03/2022
Accepted:30/03/2022

KEYWORDS
Diabetes mellitus
Complications
Mathematical model
Stability analysis

INTRODUCTION

Diabetes is a metabolic disorder characterized by the failure of the body system to normalize the amount of glucose (special sugar) in the body. This happen when the insulin secretes by the organ (pancreas) in the beta cell is not adequate or the body does not make use of the insulin produced effectively. The resulting consequence is that the body system cannot function well and glucose level in the blood goes outside normal [1], [2], [3]. The glycose - insulin regulatory system is an essential part of maintaining a healthy body. The pancreas and the lever control the production of insulin and glycose respectively, in a bid to keep the glycose level in check. Insulin secreted by the pancreas, is required by the body to allow glycose to be used as fuel in the cells. Failure to control this system will result in elevated blood sugar level leading to diabetes due to either lack of insulin or insulin resistance and is the cause of many protracted health challengess.

An uncontrolled diabetes leads to complications and complications lead to deaths. Most diabetes cases in the world are not diagnosed early at the preliminary stage. Complications developed when diabetes are not manage or detected early. There are acute and chronic complications of diabetes. The acute complications are usually severe and call for urgent medical attention while chronic complications are very slow in progression. Increasing cases of cardiovascular disease, renal failure, blindness, cognitive and psychiatric illness and infections are attributable to diabetes. Diabetes causes loss of sensory and motor function, uneven blood circulation in the feet and hand, and poor wound healing.

Majority of people living with diabetes died due to cardiovascular disease or chronic kidney disease [4], [5]. Factors such as older age, obesity, tobacco use and low- fiber food are the common risk factor of diabetes [6]. Diabetes coexist with infections. This can be seen in diabetes and tuberculosis and group B Streptococci [7], [8]. The connection between diabetes and tuberculosis slow down healing rate and increase risk of tuberculosis relapse, resistance and death [8]. Diabetes also coexist with Human Immunodeficiency Virus (HIV) thereby causing early death [9]. The number of deaths attributable to diabetes yearly worldwide is alarming. lately, there have been an increase in life expectancy around the world. Death rate caused by other leading non-communicable diseases such as Cancer, Cardiovascular disease and Stroke have been decreasing but that of diabetes is rising [10]. According to WHO, 2.8 percent of all deaths worldwide in 2010 are attributable to diabetes excluding deaths caused by cardiovascular disease and chronic kidney disease. 21 percent of coronary heart disease and 13 percent of stroke mortality worldwide are caused by diabetes [11]. Also, IDF revealed that 5 million adult deaths yearly, or 8.4 percent of all deaths worldwide are attributable to diabetes [12], [13].

In mathematics, stability theory addresses the stability of solutions of differential equations and the trajectories of the dynamical system under small pertubations of initial conditions. Stability of solution is very important in physical problems because it slight deviations from the mathematical model caused by unavoidable errors in the measurement do not have a corresponding slight effect on the solution, the mathematical equations describing the problem will not accurately predict the future outcome. The stabilty of system of differential equations is a precusor to the stability of the solution of the system.

Some mathematical models have been presented on the dynamics of diabetes and its complications in a population. The work of [14], [15], [16], [17] and [18] have proposed different models to examined the dynamics of the disease progrssion and stability of the system. In the work of [19], the method of Bellman and Coke was applied to determine the stablity of disease free equilibrium state of the model. The model are compartmentalized into diabetics with complications, diabetics with controlled sugar and diabetics without complications. The model equations were solved and

*CORRESPONDING AUTHOR |P.O. Aye | ✉patrick.aye@aaua.edu.ng

disease free equilibrum state obtained. The result obtained showed that diabetic with complications and birth rate determined the stability of the equilibrum solution of the model.

This study aimed to present a modified mathematical model for the dynamics of diabetes and its complications in a population, incorporate healthy and treatment classes and investigate the stability of the system using Bellman and Coke method.

MODEL FORMULATION

The model equations are formulated using first order differential equations. Improving on the existing work of [14], [18] and [19], a mathematical model of diabetes and its complications incorporating healthy and treatment classes was introduced. The increasing number of diabetes is influenced by the number of incidences that occur. The incidence of diabetes are caused by two factors namely unhealthy lifestyle and genetic factors of parents who have history of diabetes. The incidence of diabetes is largely due to unhealthy lifestyle such as low levels of physical activity, irregular eating patterns and other unhealthy habits such as smoking, alcoholism and glutoning.The peculiarity of the disease dynamic is that the diabetes cannot be cured but can only be manage by treatment, dieting and physical activities.

This model assumed that the healthy individual will give birth to a healthy children that will be born into healthy compartment while parent who is diabetic or have history of diabetes will give birth to children with genetic factors that will be born into susceptible compartment.The proportion of children born into healthy compartment is denoted by θ while proportion of children that are born into susceptible compartment is denoted by $1-\theta$. The population in this model is classified according to their health conditions into five compartments. They are healthy class $H(t)$, susceptible class $S(t)$, diabetic without complications class $D(t)$, diabetic with complications class $C(t)$ and diabetic with complications that undergo treatment class $T(t)$. The study also assumed that diabetes disease infections can either be acute or chronic. The *HSDCT* model is a mathematical model in the epidemiology of diabetes that takes into account lifestyle factors and genetic factors as causes of incidence of diabetes mellitus.

Assumption of the model

The models are formulated based on the following assumptions:

i. The population is compartmentalized into the following classes.
Namely: healthy class $H(t)$, susceptible class $S(t)$, diabetic without complications class $D(t)$, diabetic with complications class $C(t)$, diabetic with complications that undergo treatment class $T(t)$.

ii. Susceptible individual can move into healthy class
iii. Healthy individual can become susceptible
iv. The diabetes disease infections can either be acute or chronic
v. There can be death due to complications
vi. After successful treatment, member from the treatment class can move into diabetic without complications class
vii. The treated individual can develop complication again
viii. Incidence of the disease can occur without complications

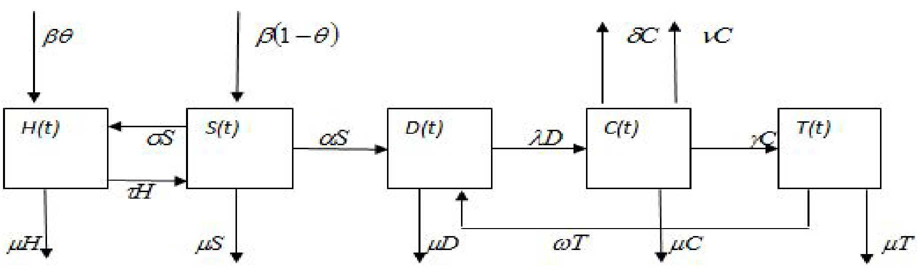

Figure 1. Schematic of the model.

Table 1. Notation and definition of variables and parameters.

Variables	Description	Variables
H(t)	*Healthy class*	*H(t)*
S(t)	*Susceptible class*	*S(t)*
D(t)	*Diabetics without complications class*	*D(t)*
C(t)	*Diabetics with complications class*	*C(t)*
T(t)	*Diabetic with complications undergoing treatment*	*T(t)*

Parameters	Description
α	*Probability rate of incidence of diabetes*
β	*Birth rate*
μ	*Natural mortality death*
τ	*Rate at which healthy individual become susceptible*
σ	*Rate at which susceptible individual become healthy*
λ	*Rate of a diabetic person developing a complications*
γ	*Rate at which diabetic with complications are treated*
v	*Rate at which diabetic patients with complication become severely disabled*
ω	*Rate at which diabetic with complications after treatment return diabetic without complications*
δ	*Mortality rate due to complications*
θ	*Proportion of children born into the healthy class*
$1-\theta$	*Proportion of children born into the susceptible class*

The model equation

$$\left.\begin{aligned}
\frac{dH}{dt} &= \beta\theta-(\mu+\tau)H+\sigma S\\
\frac{dS}{dt} &= \beta(1-\theta)+\tau H-(\mu+\alpha+\sigma)S\\
\frac{dD}{dt} &= \alpha S-\mu D-\lambda D+\omega T\\
\frac{dC}{dt} &= \lambda D-(\mu+v+\delta+\gamma)C\\
\frac{dT}{dt} &= \gamma C-(\mu+\omega)T
\end{aligned}\right\} \qquad (1)$$

STABILITY ANALYSIS

The system of equations in (1) can be represented as follows:

$$\left.\begin{aligned}
F(H,S,D,C,T) &= \frac{dH(t)}{dt} = \beta\theta-\mu H(t)-\tau H(t)+\sigma S(t)\\
F(H,S,D,C,T) &= \frac{dS(t)}{dt} = \beta(1-\theta)+\tau H(t)-\mu S(t)-\alpha S(t)-\sigma S(t)\\
F(H,S,D,C,T) &= \frac{dD(t)}{dt} = \alpha S(t)+\omega T(t)-\mu D(t)-\lambda D(t)\\
F(H,S,D,C,T) &= \frac{dC(t)}{dt} = \lambda D(t)-\mu C(t)-\delta C(t)-v C(t)-\gamma C(t)\\
F(H,S,D,C,T) &= \frac{dT(t)}{dt} = \gamma C(t)-\mu T(t)-\omega T(t)
\end{aligned}\right\} \qquad (2)$$

The Jacobian of the system in (2) is given by

$$JAC=\begin{bmatrix}\frac{\partial f}{\partial H}&\frac{\partial f}{\partial S}&\frac{\partial f}{\partial D}&\frac{\partial f}{\partial C}&\frac{\partial f}{\partial T}\\\frac{\partial g}{\partial H}&\frac{\partial g}{\partial S}&\frac{\partial g}{\partial D}&\frac{\partial g}{\partial C}&\frac{\partial g}{\partial T}\\\frac{\partial h}{\partial H}&\frac{\partial h}{\partial S}&\frac{\partial h}{\partial D}&\frac{\partial h}{\partial C}&\frac{\partial h}{\partial T}\\\frac{\partial p}{\partial H}&\frac{\partial p}{\partial S}&\frac{\partial p}{\partial D}&\frac{\partial p}{\partial C}&\frac{\partial p}{\partial T}\\\frac{\partial q}{\partial H}&\frac{\partial q}{\partial S}&\frac{\partial q}{\partial D}&\frac{\partial q}{\partial C}&\frac{\partial q}{\partial T}\end{bmatrix} \tag{3}$$

$$=\begin{bmatrix}-(\mu+\tau)&\sigma&0&0&0\\\tau&-(\mu+\alpha+\sigma)&0&0&0\\0&\alpha&-(\mu+\lambda)&0&\omega\\0&0&\lambda&-(\mu+\delta+\nu+\gamma)&0\\0&0&0&\gamma&-(\mu+\omega)\end{bmatrix} \tag{4}$$

$$(JAC-\lambda I)=|JAC-\rho I=0|$$

$$\begin{vmatrix}-(\mu+\tau)-\rho&\sigma&0&0&0\\\tau&-(\mu+\alpha+\sigma)-\rho&0&0&0\\0&\alpha&-(\mu+\lambda)&0&\omega\\0&0&\lambda&-(\mu+\delta+\nu+\gamma)&0\\0&0&0&\gamma&-(\mu+\omega)\end{vmatrix}=0 \tag{5}$$

Let $\eta=\mu+\tau,\ \pi=\mu+\alpha+\sigma,\ \varphi=\mu+\lambda,\ \kappa=\mu+\delta+\nu+\gamma,\ \varepsilon=\mu+\omega,$ we have

$$\begin{vmatrix}-\eta-\rho&\sigma&0&0&0\\\tau&-\pi-\rho&0&0&0\\0&\alpha&-\phi-\rho&0&\omega\\0&0&\lambda&-\kappa-\rho&0\\0&0&0&\gamma&-\varepsilon-\rho\end{vmatrix}=0 \tag{6}$$

$$(\eta-\rho)\begin{bmatrix}-\pi-\rho&0&0&0\\\alpha&-\phi-\rho&0&\omega\\0&\lambda&-\kappa-\rho&0\\0&0&\gamma&-\varepsilon-\rho\end{bmatrix}-\sigma\begin{bmatrix}\tau&0&0&0\\0&-\phi-\rho&0&\omega\\0&\lambda&-\kappa-\rho&0\\0&0&\gamma&-\varepsilon-\rho\end{bmatrix}=0 \tag{7}$$

$$(\eta-\rho)(-\pi-\rho)\begin{vmatrix}-\phi-\rho&0&\omega\\\lambda&-\kappa-\rho&0\\0&\gamma&-\varepsilon-\rho\end{vmatrix}-\tau\sigma\begin{vmatrix}-\phi-\rho&0&\omega\\\lambda&-\kappa-\rho&0\\0&\gamma&-\varepsilon-\rho\end{vmatrix}=0 \tag{8}$$

$$((\eta-\rho)(-\pi-\rho)-\tau\sigma)\begin{vmatrix}-\phi-\rho&0&\omega\\\lambda&-\kappa-\rho&0\\0&\gamma&-\varepsilon-\rho\end{vmatrix}=0 \tag{9}$$

$$(\eta\pi+\eta\rho+\pi\rho+\rho^2-\tau\sigma)(-\phi-\rho)(-\kappa-\rho)(-\varepsilon-\rho)=0 \tag{10}$$

Therefore we have

$$\left(\eta\pi+\eta\rho+\pi\rho+\rho^2-\tau\sigma\right)\left(\phi\rho+\phi\kappa+\kappa\rho+\rho^2\right)\left(-\varepsilon-\rho\right)=0 \quad (11)$$

Next; we investigate the stability of the system by applying the result of [20] to (11). We first present a fundamental theorem for stability analysis of the characteristic equation as follows:

Theorem (Bellman and Coke Theorem): Let $H(Z)=P\left(Z,e^Z\right)$ where P a polynomial with principal term. Suppose $H(Z)$, is separated into its real and imaginary parts as follows;

$$H(Z)=F(y)+iG(y) \quad (12)$$

If all zeros of $H(Z)$ have negative real parts, then the zeros of $F(y)$ *and* $G(y)$ are real, simple and alternate and

$$J=G^{'}(0)F(0)-G(0)F^{'}(0)>0 \quad (13)$$

Conversely, all zeros of $H(Z)$ will be in the left half plane provided that either of the following conditions is satisfied.

- All the zeros of $F(y)$ *and* $G(y)$ are real, simple, and alternate; and the inequality in (12) is satisfied
- All the zeros of $F(y)$ are real, and for each zero the relation (12) is satisfied. All the zeros of $G(y)$ are real, and for each zero, the relation (12) is satisfied

Thus, the condition for stability according to Bellman and Coke theorem is given by

$$J=G^{'}(0)F(0)-G(0)F^{'}(0)>0 \quad (14)$$

Hence the disease prevalence equilibrium state will be stable if $J>0$.

Applying Bellman and Coke theorem to the system

Let (11) take the form;

$$H(\rho)=\left(\eta\pi+\eta\rho+\pi\rho+\rho^2-\tau\sigma\right)\left(\phi\rho+\phi\kappa+\kappa\rho+\rho^2\right)\left(-\varepsilon-\rho\right)$$

Expanding and re-arranging in ascending powers of ρ we obtained

$$\begin{aligned}H(\rho)=&-\eta\pi\phi\kappa\varepsilon-\eta\pi\phi\varepsilon\rho-\eta\pi\kappa\varepsilon\rho-\eta\pi\varepsilon\rho^2-\eta\phi\kappa\varepsilon\rho-\eta\phi\varepsilon\rho-\eta\kappa\varepsilon\rho^2-\eta\varepsilon\rho^2-\eta\phi\kappa\varepsilon\rho\\&-\pi\phi\varepsilon\rho^2-\pi\kappa\varepsilon\rho-\pi\varepsilon\rho^3-\phi\kappa\varepsilon\rho-\phi\varepsilon\rho^3-\kappa\varepsilon\rho^3-\varepsilon\rho^4+\omega\kappa\tau\sigma+\phi\varepsilon\tau\sigma\rho+\kappa\varepsilon\tau\sigma\rho\\&+\varepsilon\tau\sigma\rho^2-\eta\pi\phi\kappa\rho-\eta\pi\phi\rho^2-\eta\pi\kappa\rho^2-\eta\pi\rho^3-\eta\phi\kappa\rho^2-\eta\phi\rho^3-\eta\kappa\rho^3-\eta\rho^4-\pi\phi\kappa\rho^2\\&-\pi\phi\rho^3-\pi\kappa\rho^3-\pi\rho^4-\phi\kappa\rho^3-\phi\rho^4-\kappa\rho^4-\rho^5+\phi\kappa\tau\sigma\rho+\phi\tau\sigma\rho^2+\kappa\tau\sigma\rho^2+\tau\sigma\rho^3\end{aligned} \quad (15)$$

$$\begin{aligned}H(\rho)=&-\rho^5-\eta\rho^4-\pi\rho^4-\phi\rho^4-\kappa\rho^4-\varepsilon\rho^4-\eta\pi\rho^3-\eta\phi\rho^3-\eta\kappa\rho^3-\eta\varepsilon\rho^3-\pi\phi\rho^3\\&-\pi\kappa\rho^3-\pi\varepsilon\rho^3-\phi\kappa\rho^3-\phi\varepsilon\rho^3-\kappa\varepsilon\rho^3+\tau\sigma\rho^3-\eta\pi\phi\rho^2-\eta\pi\kappa\rho^2-\eta\pi\varepsilon\rho^2-\eta\phi\kappa\rho^2\\&-\eta\phi\varepsilon\rho^2-\eta\kappa\varepsilon\rho^2-\pi\phi\kappa\rho^2-\pi\phi\varepsilon\rho^2-\pi\kappa\varepsilon\rho^2-\phi\kappa\varepsilon\rho^2+\phi\tau\sigma\rho^2+\kappa\tau\sigma\rho^2+\varepsilon\tau\sigma\rho^2\\&-\eta\pi\phi\kappa\rho-\eta\pi\phi\varepsilon\rho-\eta\pi\kappa\varepsilon\rho-\eta\phi\kappa\varepsilon\rho-\pi\phi\kappa\varepsilon\rho+\phi\kappa\tau\sigma\rho+\phi\varepsilon\tau\sigma\rho+\kappa\varepsilon\tau\sigma\rho-\eta\pi\phi\kappa\varepsilon\\&+\phi\kappa\varepsilon\tau\sigma\end{aligned}$$

$$\begin{aligned}H(\rho)=&-\rho^5-\left(\eta+\pi+\phi+\kappa+\varepsilon\right)\rho^4-\begin{pmatrix}\eta\pi+\eta\phi+\eta\kappa+\eta\varepsilon\\+\pi\phi+\pi\kappa+\pi\varepsilon+\phi\kappa\\+\phi\varepsilon+\kappa\varepsilon-\tau\sigma\end{pmatrix}\rho^3-\\&\begin{pmatrix}\eta\pi\phi+\eta\pi\kappa+\eta\pi\varepsilon+\eta\phi\kappa\\+\eta\phi\varepsilon+\eta\kappa\varepsilon+\pi\phi\kappa+\pi\phi\varepsilon\\+\pi\kappa\varepsilon+\phi\kappa\varepsilon-\phi\tau\sigma-\kappa\tau\sigma-\varepsilon\tau\sigma\end{pmatrix}\rho^2-\begin{pmatrix}\eta\pi\phi\kappa+\eta\pi\phi\varepsilon+\eta\pi\kappa\varepsilon+\eta\phi\kappa\varepsilon\\+\pi\phi\kappa\varepsilon-\phi\kappa\tau\sigma-\phi\varepsilon\tau\sigma-\kappa\varepsilon\tau\sigma\rho\end{pmatrix}\rho^2\\&-\eta\pi\phi\kappa\varepsilon+\phi\kappa\varepsilon\tau\sigma\end{aligned} \quad (16)$$

We now set $\rho = iw$ in equation (16) to have

$$H(\rho) = -(iw)^5 - (\eta+\pi+\phi+\kappa+\varepsilon)(iw)^4 - \begin{pmatrix} \eta\pi+\eta\phi+\eta\kappa+\eta\varepsilon \\ +\pi\phi+\pi\kappa+\pi\varepsilon+\phi\kappa \\ +\phi\varepsilon+\kappa\varepsilon-\tau\sigma \end{pmatrix}(iw)^3 - \begin{pmatrix} \eta\pi\phi+\eta\pi\kappa+\eta\pi\varepsilon+\eta\phi\kappa \\ +\eta\phi\varepsilon+\eta\kappa\varepsilon+\pi\phi\kappa+\pi\phi\varepsilon \\ +\pi\kappa\varepsilon+\phi\kappa\varepsilon-\phi\tau\sigma-\kappa\tau\sigma-\varepsilon\tau\sigma \end{pmatrix}(iw)^2 - \begin{pmatrix} \eta\pi\phi\kappa+\eta\pi\phi\varepsilon+\eta\pi\kappa\varepsilon+\eta\phi\kappa\varepsilon \\ +\pi\phi\kappa\varepsilon-\phi\kappa\tau\sigma-\phi\varepsilon\tau\sigma-\kappa\varepsilon\tau\sigma\rho \end{pmatrix}(iw) - \eta\pi\phi\kappa\varepsilon+\phi\kappa\varepsilon\tau\sigma \tag{17}$$

$$H(\rho) = -w^5 i - (\eta+\pi+\phi+\kappa+\varepsilon)w^4 i + \begin{pmatrix} \eta\pi+\eta\phi+\eta\kappa+\eta\varepsilon \\ +\pi\phi+\pi\kappa+\pi\varepsilon+\phi\kappa \\ +\phi\varepsilon+\kappa\varepsilon-\tau\sigma \end{pmatrix} w^3 i + \begin{pmatrix} \eta\pi\phi+\eta\pi\kappa+\eta\pi\varepsilon+\eta\phi\kappa \\ +\eta\phi\varepsilon+\eta\kappa\varepsilon+\pi\phi\kappa+\pi\phi\varepsilon \\ +\pi\kappa\varepsilon+\phi\kappa\varepsilon-\phi\tau\sigma-\kappa\tau\sigma-\varepsilon\tau\sigma \end{pmatrix} w^2 i - \begin{pmatrix} \eta\pi\phi\kappa+\eta\pi\phi\varepsilon+\eta\pi\kappa\varepsilon+\eta\phi\kappa\varepsilon \\ +\pi\phi\kappa\varepsilon-\phi\kappa\tau\sigma-\phi\varepsilon\tau\sigma-\kappa\varepsilon\tau\sigma\rho \end{pmatrix} wi - \eta\pi\phi\kappa\varepsilon+\phi\kappa\varepsilon\tau\sigma \tag{18}$$

Separating (18) into real and imaginary parts we have

$$H(\rho) = (\eta+\pi+\phi+\kappa+\varepsilon)w^4 i + \begin{pmatrix} \eta\pi+\eta\phi+\eta\kappa+\eta\varepsilon \\ +\pi\phi+\pi\kappa+\pi\varepsilon+\phi\kappa \\ +\phi\varepsilon+\kappa\varepsilon-\tau\sigma \end{pmatrix} w^3 i + \begin{pmatrix} \eta\pi\phi+\eta\pi\kappa+\eta\pi\varepsilon+\eta\phi\kappa \\ +\eta\phi\varepsilon+\eta\kappa\varepsilon+\pi\phi\kappa+\pi\phi\varepsilon \\ +\pi\kappa\varepsilon+\phi\kappa\varepsilon-\phi\tau\sigma-\kappa\tau\sigma-\varepsilon\tau\sigma \end{pmatrix} w^2 i - \eta\pi\phi\kappa\varepsilon+\phi\kappa\varepsilon\tau\sigma - w^5 i - \begin{pmatrix} \eta\pi\phi\kappa+\eta\pi\phi\varepsilon+\eta\pi\kappa\varepsilon+\eta\phi\kappa\varepsilon \\ +\pi\phi\kappa\varepsilon-\phi\kappa\tau\sigma-\phi\varepsilon\tau\sigma-\kappa\varepsilon\tau\sigma\rho \end{pmatrix} wi \tag{19}$$

We then apply the result of [17] to analyze the stability or otherwise. Resolving (19) into real and imaginary parts we have ; $H(iw) = F(w) + iG(w)$, where $F(w)$ and $G(w)$ are given respectively by

$$F(w) = -(\eta+\pi+\phi+\kappa+\varepsilon)w^4 + \begin{pmatrix} \eta\pi\phi+\eta\pi\kappa+\eta\pi\varepsilon+\eta\phi\kappa \\ +\eta\phi\varepsilon+\eta\kappa\varepsilon+\pi\phi\kappa+\pi\phi\varepsilon \\ +\pi\kappa\varepsilon+\phi\kappa\varepsilon-\phi\tau\sigma-\kappa\tau\sigma-\varepsilon\tau\sigma \end{pmatrix} w^2 - \eta\pi\phi\kappa\varepsilon+\phi\kappa\varepsilon\tau\sigma \tag{20}$$

$$G(w) = -w^5 + \begin{pmatrix} \eta\pi+\eta\phi+\eta\kappa+\eta\varepsilon \\ +\pi\phi+\pi\kappa+\pi\varepsilon+\phi\kappa \\ +\phi\varepsilon+\kappa\varepsilon-\tau\sigma \end{pmatrix} w^3 - \begin{pmatrix} \eta\pi\phi\kappa+\eta\pi\phi\varepsilon+\eta\pi\kappa\varepsilon+\eta\phi\kappa\varepsilon \\ +\pi\phi\kappa\varepsilon-\phi\kappa\tau\sigma-\phi\varepsilon\tau\sigma-\kappa\varepsilon\tau\sigma\rho \end{pmatrix} w \tag{21}$$

Differentiating (20) and (21) with respect to w, we have

$$F'(w) = -4w^3(\eta+\pi+\phi+\kappa+\varepsilon) + 2w \begin{pmatrix} \eta\pi\phi+\eta\pi\kappa+\eta\pi\varepsilon+\eta\phi\kappa \\ +\eta\phi\varepsilon+\eta\kappa\varepsilon+\pi\phi\kappa+\pi\phi\varepsilon \\ +\pi\kappa\varepsilon+\phi\kappa\varepsilon-\phi\tau\sigma-\kappa\tau\sigma-\varepsilon\tau\sigma \end{pmatrix} \tag{22}$$

$$G'(w) = -5w^4 + 3w^2 \begin{pmatrix} \eta\pi+\eta\phi+\eta\kappa+\eta\varepsilon \\ +\pi\phi+\pi\kappa+\pi\varepsilon+\phi\kappa \\ +\phi\varepsilon+\kappa\varepsilon-\tau\sigma \end{pmatrix} - \begin{pmatrix} \eta\pi\phi\kappa+\eta\pi\phi\varepsilon+\eta\pi\kappa\varepsilon+\eta\phi\kappa\varepsilon \\ +\pi\phi\kappa\varepsilon-\phi\kappa\tau\sigma-\phi\varepsilon\tau\sigma-\kappa\varepsilon\tau\sigma\rho \end{pmatrix} \tag{23}$$

Setting $w = 0$ in (20), (21), (22) and (23) gives

$$F(0) = -\eta\pi\phi\kappa\varepsilon + \phi\kappa\varepsilon\tau\sigma \quad (24)$$

$$G(0) = 0 \quad (25)$$

$$F'(0) = 0 \quad (26)$$

$$G'(0) = -(\eta\pi\phi\kappa + \eta\pi\phi\varepsilon + \eta\pi\kappa\varepsilon + \eta\phi\kappa\varepsilon + \pi\phi\kappa\varepsilon - \phi\kappa\tau\sigma - \phi\varepsilon\tau\sigma - \kappa\varepsilon\tau\sigma) \quad (27)$$

According to the result of [17] , the condition for stability is given by

$$J = G'(0)F(0) - G(0)F'(0) > 0 \quad (28)$$

Now, since $G(0) = F'(0) = 0$,we have $G'(0)F(0) \neq 0$, thus we have

$$J = G'(0)F(0) = (\eta\pi\phi\kappa\varepsilon + \phi\kappa\varepsilon\tau\sigma)\begin{pmatrix}\eta\pi\phi\kappa + \eta\pi\phi\varepsilon + \eta\pi\kappa\varepsilon + \eta\phi\kappa\varepsilon \\ +\pi\phi\kappa\varepsilon - \phi\kappa\tau\sigma - \phi\varepsilon\tau\sigma - \kappa\varepsilon\tau\sigma\end{pmatrix} \quad (29)$$

Now if J in (29) evaluate to positive (+ve) value for values of the parameters, then the system is stable else unstable.

NUMERICAL COMPUTATION OF J VALUE

The numerical computation of the J value was done using hypothetical value from Table 2 below.

Table 2. Hypothetical values adopted from Aye et al.[21].

S/N	Parameter	Value
1	δ	0.05
2	μ	0.02
3	λ	0.08
4	σ	0.008
5	α	0.05
6	ν	0.05
7	τ	0,04
8	γ	0.08
9	ω	0.08

The various parameters in equation (29) is calculated as follows

$\eta\pi\phi\kappa\varepsilon = 0.0000126$

$\phi\kappa\varepsilon\tau\sigma = 0.000000448$

$\eta\pi\phi\kappa = 0.000126$

$\eta\pi\phi\varepsilon = 0.000063$

$\eta\pi\kappa\varepsilon = 0.00018$

$\eta\phi\kappa\varepsilon = 0.000084$

$\pi\phi\kappa\varepsilon = 0.00021$

$\phi\kappa\tau\sigma = 0.0000448$

$\phi\varepsilon\tau\sigma = 0.0000224$

$\kappa\varepsilon\tau\sigma = 0.000064$

$$J = (0.0000126 - 0.00000448)\begin{pmatrix}0.000126 + 0.000063 + 0.00018 + 0.000084 \\ +0.00021 - 0.0000448 - 0.0000224 - 0.000064\end{pmatrix} \quad (30)$$

$$= 0.000000043 > 0$$

From (30), J evaluated to positive value. This means that the system is very stable.

CONCLUSION

This study proposed a mathematical model for the dynamics of diabetes mellitus and its complications in a population by incorporating treatment and positive lifestyle as control measure. The model is a linear system of ordinary differential equations. The stability of the system was analyzed using Bellman and Coke Method and the system was found to be stable. This result established that the method of Bellman and Coke can be adequately apply to analyze the stability of epidemilogical linear models.

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