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ORIGINAL ARTICLE

Comparison among several numerical methods for solving Volterra-Fredholm integro-differential equations

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ABSTRACT – The Volterra-Fredholm integro-differential equations (VFIDEs) are complicated to solve analyticlly. In many cases, they required to obtain the approximate solutions. Therefore, the numerical methods are used to introduce approximate solutions for this types of equations. In this paper, we studied the comparison among several numerical methods for solving special orders of such types of equations. The comparison showed that, these numerical methods are acceptable and reliable numerical techniques VFIDEs of the second kind.

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INTRODUCTION

There are many papers that have studied integral and integro-differential equations by several types of numerical methods in order to introduce an approximate numerical solutions for such type of equations. The author in [1] used Bernstein Polynomials method for solving Volterra-Fredholm integral equations of the second kind. While, in [2] Boubaker polynomials method has been applied to introduce an approximate solution for Volterra-Fredholm integral equation of the second kind. The same method has been used in [3] to find an approximate solution for the second kind Volterra and Fredholm integral equatins. In [4] proposed the variation formulation by using the basis Boubaker polynomials for approximate the solution of Volterra-Fredholm integro-differential equation via variational formulation method. Furthermore, new collocation method, which is based on Boubaker polynomials was introduced in [5] for approximate solutions of mixed linear integro-differential-difference equations under the mixed conditions. While, in [6] Boubaker polynomials collocation approach has been used for solving a system of nonlinear Volterra-Fredholm integral equations of the second kind. However, Boubaker polynomials method has been used in [7] to find an approximate solution for the initial value problem of the nonlinear high order Volterra and Fredholm integro-differental equations of the second kind. In addition, Bernstein polynomials method was applied in [8] for solving Volterra-Fredholm integro-differential equations of the second kind. Then, in [9] integral collocation approximation methods which are power series, Chebyshev polynomials and Legendre's polynomials have been applied to solve high order linear Volterra-Fredholm integro-differential equations. The Least-Squares method and Laguerre polynomials have been used in [10] to present numerical solution for mixed integro-differential equations. On the other hand, a comparsion between the Variational Iteration method and Trapezoidal Rule for solving the linear Volterra and Fredholm integro-differential equations have been presented in [11]. While, a comparison between Touchard polynomials method and Bernstein polynomials method for solving nonlinear Volterra integral equation are presented in [12]. In the same year, the comparison between Touchard polynomials method and Bernstein polynomials method has been presented in [13] for solving the linear Fredholm integral equation of the second kind. Moreover, a comparison among the Finite difference method (FDM), Spectral method and Wavelet Galerkin method (WGM) has been presented in [14] for solving Partial differential equations (PDEs). In this paper, we studied the numerical comparison among several numerical methods for solving special orders Volterra-Fredholm integro-differential equations of the second kind. This paper is organized as follows: Boubaker polynomials method has been presented in section 2. While the matrix representation of the Boubaker polynomials method has been stated in section 3. Section 4 is illustrated by Laguerre polynomials method. Section 5 followed by some numerical examples. The comparison among the numerical methods has been presented in section 6 to section 9. Finally, the conclusion and future research scope of this paper are drown in section 10.

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BOUBAKER POLYNOMIALS METHOD

The Boubaker Polynomials method has been initially defined with the physical study in order to get an analytical solution for the heat equation [15]. The Boubaker polynomials have been defined in [16] as follows:

$$\Psi_j(\rho)=\sum_{i=0}^{[j/2]}(-1)^i\binom{j-i}{i}\frac{j-4i}{j-i}\rho^{j-2i}, j\geq 1. \tag{1}$$

The first eight members of Boubaker polynomials are:

$$\Psi_0(\rho)=1$$

$$\Psi_1(\rho)=\rho$$

$$\Psi_2(\rho)=\rho^2+2$$

$$\Psi_3(\rho)=\rho^3+\rho$$

$$\Psi_4(\rho)=\rho^4-2$$

$$\Psi_5(\rho)=\rho^5-\rho^3-3\rho$$

$$\Psi_6(\rho)=\rho^6-2\rho^4-3\rho^2+2$$

$$\Psi_7(\rho)=\rho^7-3\rho^5-2\rho^3+5\rho.$$

The three-term recurrence relation is given by

$$\Psi_l(\rho)=\rho\Psi_{l-1}(\rho)-\Psi_{l-2}(\rho),\quad l\geq 2.$$

MATRIX FORMULATION FOR BOUBAKER POLYNOMIALS METHOD

In this section, we presented the matrix formulation for the Boubaker polynomials method. The Boubaker polynomial can be written as a linear combination of Boubaker basis functions as:

$$\Psi_j(\rho)=\pi_0\Psi_0(\rho)+\pi_1\Psi_1(\rho)+\pi_2\Psi_2(\rho)+\cdots+\pi_j\Psi_j(\rho), \tag{2}$$

where $\pi_r, r=0,1,2,\ldots,j$ are the unknown cofficients. Equation (2) can be written as a dot scalar of two vectors:

$$\Psi_j(\rho)=\begin{bmatrix}\Psi_0(\rho) & \Psi_1(\rho) & \Psi_2(\rho) & \cdots & \Psi_j(\rho)\end{bmatrix}\cdot\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}. \tag{3}$$

Equation (3) can also be written:

$$\Psi_j(\rho)=\begin{bmatrix}1 & \rho & \rho^2 & \cdots & \rho^j\end{bmatrix}\cdot\begin{bmatrix}\beta_{00} & \beta_{01} & \beta_{02} & \cdots & \beta_{0j}\\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1j}\\ 0 & 0 & \beta_{22} & \cdots & \beta_{2j}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & 0 & \beta_{jj}\end{bmatrix}\cdot\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}, \tag{4}$$

where β's are the cofficients of the power basis that can be used to determine the respective Boubaker polynomial.

LAGUERRE POLYNOMIALS METHOD

The Laguerre polynomials is defined as follow [17]:

$$H_k(\mu)=\sum_{e=0}^{k}\frac{(-1)^e}{e!}\binom{k}{e}\mu^e, k=0,1,2,\dots,n \text{ and } \mu\in[0,\infty) \tag{5}$$

where k is the degree and e is the index of the Laguerre polynomials. The first five Laguerre polynomials from equation (5) are defined as:

$$H_0(\mu)=1,$$
$$H_1(\mu)=1-\mu,$$
$$H_2(\mu)=\frac{1}{2}(2-4\mu+\mu^2),$$
$$H_3(\mu)=\frac{1}{6}(6-18\mu+9\mu^2-\mu^3),$$
$$H_4(\mu)=\frac{1}{24}(24-96\mu+72\mu^2-16\mu^3+\mu^4).$$

The Laguerre polynomials can be written as a linear combination of Laguerre basis functions in the form:

$$H_k(\mu)=s_0H_0(\mu)+s_1H_1(\mu)+s_2H_2(\mu)+\dots+s_kH_k(\mu), \tag{6}$$

where $s_0,s_1,s_2,\dots,s_k$ are the unknown coefficients. Equation (6) can be written as a dot scalar of two vectors:

$$H_k(\mu)=\begin{bmatrix}H_0(\mu) & H_1(\mu) & H_2(\mu) & \cdots & H_k(\mu)\end{bmatrix}\begin{bmatrix}s_0\\ s_1\\ s_2\\ \vdots\\ s_k\end{bmatrix}. \tag{7}$$

Equation (7) can also written as:

$$H_k(\mu)=\begin{bmatrix}1 & \mu & \mu^2 & \cdots & \mu^k\end{bmatrix}\cdot\begin{bmatrix}\gamma_{00} & \gamma_{01} & \gamma_{02} & \cdots & \gamma_{0k}\\ 0 & \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1k}\\ 0 & 0 & \gamma_{22} & \cdots & \gamma_{2k}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & 0 & \gamma_{kk}\end{bmatrix}\cdot\begin{bmatrix}s_0\\ s_1\\ s_2\\ \vdots\\ s_k\end{bmatrix}, \tag{8}$$

where γ's are the cofficients of the power basis that can be used to determine the respective Laguerre polynomial.

Next, to solve Volterra-Fredholm integro-differential equations of the second kind (VFIDE2K) by using Boubaker polynomials method or Laguerre polynomials method, we apply the following proceduers. Considere the VFIDE2K given in equation (9) as:

$$\sum_{k=0}^{m}\sigma_k(z)F^k(z)=\phi(z)+\eta_1\int_a^z w_1(z,t)F(t)dt+\eta_2\int_a^b w_2(z,t)F(t)dt \tag{9}$$

where the initial condition $F^k(a)=F_k$, $k=1,2,\dots,m$, for each $a,b,\eta_1,\eta_2\in R$, $\phi(z)$, $w_1(z,t)$, $w_2(z,t)$ and $\sigma_k(z)$, $k=1,2,\dots,m$ are known functions that have derivative on the interval $[0,1]$ and $F(z)$ is the unknown function which will be determined. We note that $\sigma_k(z)\neq 0$.

Suppose that $F(z)=\Psi_j(\rho)$, then

$$F(z)=\pi_0\Psi_0(\rho)+\pi_1\Psi_1(\rho)+\pi_2\Psi_2(\rho)+\cdots+\pi_j\Psi_j(\rho), \tag{10}$$

where $\Psi_j(\rho)$ is the Boubaker basis polynomial and $\pi_0,\pi_1,\pi_2,\ldots,\pi_j$ are the Boubaker coefficients which will be determined.

Equation (10) can be written as a dot product:

$$F(z)=\begin{bmatrix}\Psi_0(\rho) & \Psi_1(\rho) & \Psi_2(\rho) & \cdots & \Psi_j(\rho)\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}. \tag{11}$$

Equation (11) can be further written as:

$$F(z)=\begin{bmatrix}1 & \rho & \rho^2 & \cdots & \rho^j\end{bmatrix}\begin{bmatrix}\beta_{00} & \beta_{01} & \beta_{02} & \cdots & \beta_{0j}\\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1j}\\ 0 & 0 & \beta_{22} & \cdots & \beta_{2j}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \beta_{jj}\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}. \tag{12}$$

Substituting equation (12) into equation (9) we get:

$$\sum_{k=0}^{m}\sigma_k(z)\left[\begin{bmatrix}\Psi_0(\rho) & \Psi_1(\rho) & \Psi_2(\rho) & \cdots & \Psi_j(\rho)\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}\right]^k=\phi(z)$$

$$+\eta_1\int_a^z w_1(z,t)\begin{bmatrix}\Psi_0(\rho) & \Psi_1(\rho) & \Psi_2(\rho) & \cdots & \Psi_j(\rho)\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}dt$$

$$+\eta_2\int_a^b w_2(z,t)\begin{bmatrix}\Psi_0(\rho) & \Psi_1(\rho) & \Psi_2(\rho) & \cdots & \Psi_j(\rho)\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}dt \tag{13}$$

Now, applying equation (12) into equation (13) we get:

$$\sum_{k=0}^{m}\sigma_k(z)\left[\begin{bmatrix}\Psi_0(\rho) & \Psi_1(\rho) & \Psi_2(\rho) & \cdots & \Psi_j(\rho)\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}\right]^k = \phi(z)$$

$$+\eta_1\int_a^z w_1(z,t)\begin{bmatrix}1 & \rho & \rho^2 & \cdots & \rho^j\end{bmatrix}\begin{bmatrix}\beta_{00} & \beta_{01} & \beta_{02} & \cdots & \beta_{0j}\\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1j}\\ 0 & 0 & \beta_{22} & \cdots & \beta_{2j}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \beta_{jj}\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}dt$$

$$+\eta_2\int_a^b w_2(z,t)\begin{bmatrix}1 & \rho & \rho^2 & \cdots & \rho^j\end{bmatrix}\begin{bmatrix}\beta_{00} & \beta_{01} & \beta_{02} & \cdots & \beta_{0j}\\ 0 & \beta_{11} & \beta_{12} & \cdots & \beta_{1j}\\ 0 & 0 & \beta_{22} & \cdots & \beta_{2j}\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \beta_{jj}\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \vdots\\ \pi_j\end{bmatrix}dt \tag{14}$$

After computing the integrations on the right hand side of equation (14), the unknown coefficients $\pi_0,\pi_1,\pi_2,\ldots,\pi_j$ are found by selecting z_i in the interval $[0,1]$ which can be calculated in terms of the formula $z_i = a + id$ where $d = \dfrac{b-a}{j}$, $j = 0,1,2,\ldots$. After that, a system of algebraic equations which can be solved by using Gauss Elimination to determine the values of these unknown coefficients.

The above procedure can be applied to Laguerre polynomials method.

NUMERICAL EXAMPLES

In this section, three examples are solved by using Boubaker polynomials method. The computations associated with the examples were performed by using MATLAB. Before we starte introducing the results, we presented the following observations:

Yapp = Approximate solution,
n = Degree of the polynomials,
L-Ps = Laguerre polynomials method,
T-Ps = Touchard polynomials method,
L.S.E = Least Square Error, Error = $\sum_{k=1}^{m}\left(y_{Exact}(z) - y_{Approximation}(z)\right)^2$.

Example 1[8] Consider the following VFIDE2K given as:

$F'(z) = 2e^z - 2 + \int_0^z F(t)dt + \int_0^1 F(t)dt$, with the initial condition $F(0) = 0$ and the exact solution is $F(z) = ze^z$.

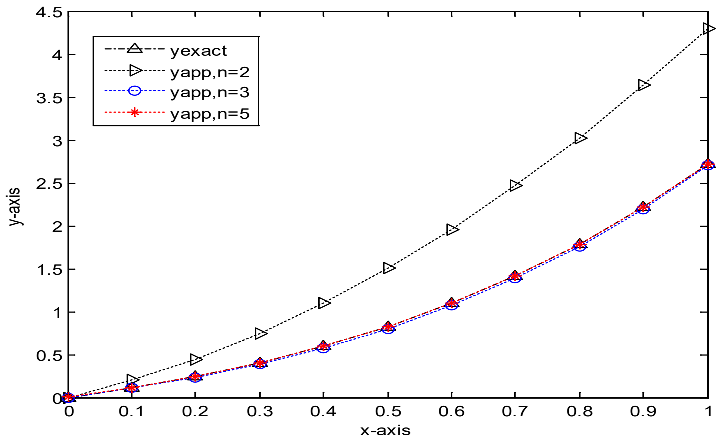

Figure 1. Exact and approximate solutions for Example 1 using Boubaker polynomials method.

Table 1. Numerical results for Example 1 using Boubaker polynomials method.

z	Exact Solution	Yapp(n=2)	Error(n=2)	Yapp(n=3)	Error(n=3)	Yapp(n=5)	Error(n=5)
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1105	0.1976	0.0076	0.1070	0.0000	0.1105	0.0000
0.2	0.2443	0.4467	0.0410	0.2344	0.0001	0.2442	0.0000
0.3	0.4050	0.7474	0.1173	0.3882	0.0003	0.4049	0.0000
0.4	0.5967	1.0997	0.2529	0.5742	0.0005	0.5966	0.0000
0.5	0.8244	1.5035	0.4612	0.7984	0.0007	0.8242	0.0000
0.6	1.0933	1.9588	0.7492	1.0668	0.0007	1.0931	0.0000
0.7	1.4096	2.4657	1.1153	1.3852	0.0006	1.4095	0.0000
0.8	1.7804	3.0241	1.5468	1.7596	0.0004	1.7803	0.0000
0.9	2.2136	3.6341	2.0178	2.1959	0.0003	2.2135	0.0000
1.0	2.7183	4.2957	2.4882	2.7002	0.0003	2.7181	0.0000

Example 2[9] Consider the following VFID2K given as:

$F''(z) = -8 + 6z - 3z^2 + z^3 + \int_0^z F(t)dt + \int_{-1}^{1} (1-2zt)F(t)dt, -1 \le z \le 1$, with the initial conditions $F(0) = 2$ and $F'(0) = 6$. The exact solution is $F(z) = 2 + 6z - 3z^2$.

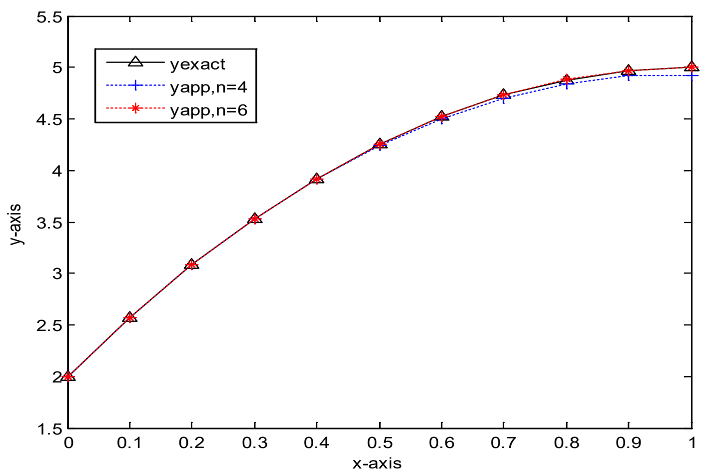

Figure 2. Exact and approximate solutions for Example 2 using Boubaker polynomials method.

Table 2. Numerical results for Example 2 using Boubaker polynomials method.

z	Exact Solution	Yapp(n=4)	Error(n=4)	Yapp(n=6)	Error(n=6)
0.0	2.0000	2.0000	0.0000	2.0000	0.0000
0.1	2.5700	2.5699	0.0000	2.5700	0.0000
0.2	3.0800	3.0794	0.0000	3.0800	0.0000
0.3	3.5300	3.5280	0.0000	3.5300	0.0000
0.4	3.9200	3.9152	0.0000	3.9201	0.0000
0.5	4.2500	4.2407	0.0001	4.2501	0.0000
0.6	4.5200	4.5039	0.0003	4.5202	0.0000
0.7	4.7300	4.7044	0.0007	4.7303	0.0000
0.8	4.8800	4.8417	0.0015	4.8804	0.0000
0.9	4.9700	4.9154	0.0030	4.9705	0.0000
1.0	5.0000	4.9249	0.0056	5.0006	0.0000

Example 3[9] Consider the following VFIDE2K given as:

$F'''(z)=\frac{z^2}{2}+\int_0^z F(t)dt+\int_{-\pi}^{\pi} zF(t)dt$, with the initial conditions $F(0)=F'(0)=-F''(0)=1$. The exact solution is $F(z)=z+\cos(z)$.

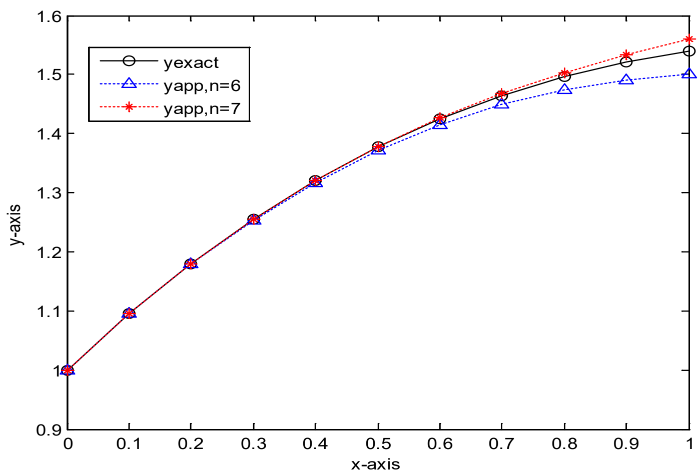

Figure 3. Exact and approximate solutions for Example 3 using Boubaker polynomials method.

Table 3. Numerical results for Example 3 using Boubaker polynomials method.

z	Exact Solution	Yapp (n=6)	Error (n=6)	Yapp (n=7)	Error (n=7)
0.0	1.0000	1.0000	0.0000	1.0000	0.0000
0.1	1.0950	1.0948	0.0000	1.0950	0.0000
0.2	1.1801	1.1793	0.0000	1.1801	0.0000
0.3	1.2553	1.2534	0.0000	1.2554	0.0000
0.4	1.3211	1.3172	0.0000	1.3213	0.0000
0.5	1.3776	1.3710	0.0000	1.3782	0.0000
0.6	1.4253	1.4148	0.0001	1.4269	0.0000
0.7	1.4648	1.4490	0.0003	1.4682	0.0000
0.8	1.4967	1.4741	0.0005	1.5031	0.0000
0.9	1.5216	1.4907	0.0010	1.5329	0.0001
1.0	1.5403	1.4997	0.0016	1.5591	0.0004

Example 1 has been solved agian using Laguerre polynomials method and the results are presented in Figure 4 and Table 4.

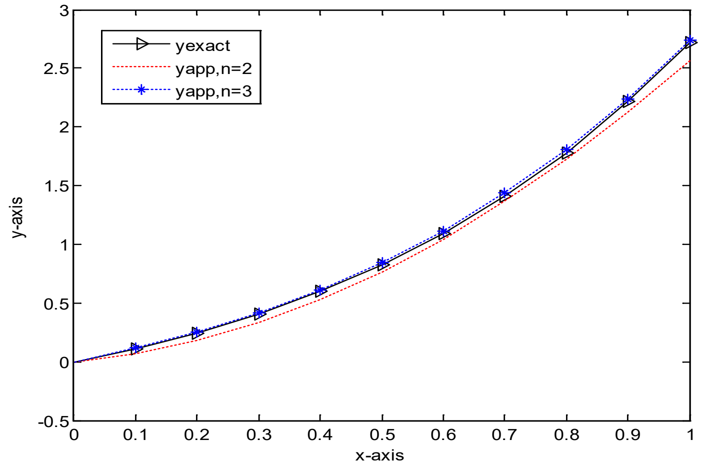

Figure 4. Exact and approximate solutions for Example 1 using Laguerre polynomials method.

Table 4. Numerical results for Example 1 using Laguerre polynomials method.

z	Exact Solution	Yapp (n=2)	Error (n=2)	Yapp (n=3)	Error (n=3)
0.0	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1105	0.0705	0.0016	0.1192	0.0001
0.2	0.2443	0.1824	0.0038	0.2572	0.0002
0.3	0.4050	0.3358	0.0048	0.4200	0.0002
0.4	0.5967	0.5305	0.0044	0.6134	0.0003
0.5	0.8244	0.7667	0.0033	0.8431	0.0004
0.6	1.0933	1.0443	0.0024	1.1150	0.0005
0.7	1.4096	1.3633	0.0021	1.4349	0.0006
0.8	1.7804	1.7237	0.0032	1.8087	0.0008
0.9	2.2136	2.1255	0.0078	2.2421	0.0008
1.0	2.7183	2.5688	0.0224	2.7409	0.0005

Example 3 has been solved again using Laguerre polynomials method and the results are given in Figure 5 and Table 5.

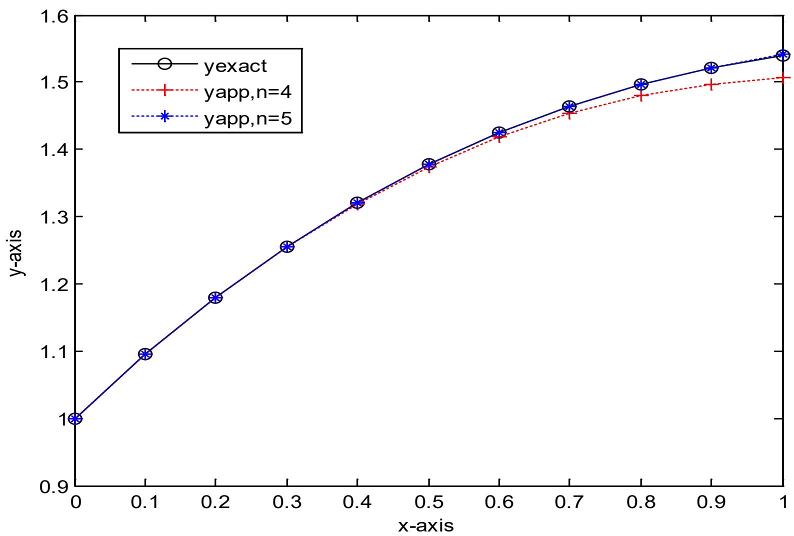

Figure 5. Exact and approximate solutions for Example 3 using Laguerre polynomials method.

Table 5. Numerical results for Example 3 using Laguerre polynomials method.

z	Exact Solution	Yapp (n=4)	Error (n=4)	Yapp (n=5)	Error (n=5)
0.0	1.0000	1.0000	0.0000	1.0000	0.0000
0.1	1.0950	1.0950	0.0000	1.0950	0.0000
0.2	1.1801	1.1799	0.0000	1.1800	0.0000
0.3	1.2553	1.2547	0.0000	1.2552	0.0000
0.4	1.3211	1.3195	0.0000	1.3208	0.0000
0.5	1.3776	1.3742	0.0000	1.3771	0.0000
0.6	1.4253	1.4192	0.0000	1.4245	0.0000
0.7	1.4648	1.4546	0.0001	1.4638	0.0000
0.8	1.4967	1.4806	0.0003	1.4957	0.0000
0.9	1.5216	1.4976	0.0006	1.5211	0.0000
1.0	1.5403	1.5060	0.0012	1.5413	0.0000

Next, the comparsions among several numerical methods have been studied and presented in the following sections.

COMPARISON BETWEEN BOUBAKER POLYNOMIALS METHOD, BERNSTEIN POLYNOMIALS METHOD AND LAGUERRE POLYNOMIALS METHOD

In this section, the comparsion have done for Example 1 between Boubaker polynomials method, Bernstein polynomials method and Laguerre polynomials method and presented in Table 6. The comparsion showed that the results are approximatly the same between Boubaker polynomials method and Bernstein polynomials method. However, for the results of Laguerre polynomials method, there is a little difference with the results of the other methods and with the exact solution while that its degree is equal to 3. Therefore, these numerical methods are efficient and accurate to estimate the solution of Example 1.

Table 6. Comparison of the numerical results for Example 1.

z	Exact Solution	Boubaker polynomials (n=5)	Bernestein polynomials (n=5) [8]	Laguerre Polynomials (n=3)
0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.1105	0.1105	0.1105	0.1192
0.2	0.2443	0.2442	0.2443	0.2572
0.3	0.4050	0.4049	0.4050	0.4200
0.4	0.5967	0.5966	0.5968	0.6134
0.5	0.8244	0.8242	0.8244	0.8431
0.6	1.0933	1.0931	1.0933	1.1150
0.7	1.4096	1.4095	1.4097	1.4349
0.8	1.7804	1.7803	1.7805	1.8087
0.9	2.2136	2.2135	2.2137	2.2421
1.0	2.7183	2.7181	2.7184	2.7409

COMPARISON BETWEEN BOUBAKER POLYNOIALS METHOD, TOUCHARD POLYNOMAILS METHOD AND LAGUERRE POLYNOMIALS METHOD

In this section, the comparsion has been studied for Example 3 and presented in Table 7. The comparison showed that when the degree of the Touchard polynomials method is five, then the approximate solution is converged to the exact solution approximately, while in the Boubaker polynomials method converges to the exact solution when its degree is seven. Furthermore, the Laguerre polynomials method had converged when its degree is equale to 3. Therefore, the results of these numerical methods are approximatly the same with the exact solution and between each other even with their degrees are different.

Table 7. Comparison of the numerical results for Example 3.

z	Exact solution	Boubaker Polynomials (n=7)	Touchard polynomials (n=5) [18]	Laguerre polynomials (n=3)
0.0	1.0000	1.0000	1.0000	1.0000
0.1	1.0950	1.0950	1.0950	1.0950
0.2	1.1801	1.1801	1.1800	1.1800
0.3	1.2553	1.2554	1.2552	1.2552
0.4	1.3211	1.3213	1.3207	1.3208
0.5	1.3776	1.3782	1.3770	1.3771
0.6	1.4253	1.4269	1.4243	1.4245
0.7	1.4648	1.4682	1.4635	1.4638
0.8	1.4967	1.5031	1.4951	1.4957
0.9	1.5216	1.5329	1.5203	1.5211
1.0	1.5403	1.5591	1.5401	1.5413

COMPARISON BETWEEN BOUBAKER POLYNOMIALS AND POWER SERIES, CHEBYSHEV POLYNOMIALS AND LEGENDER'S POLYNOMIALS

This section contains the comparsions for Example 2 and Example 3 and presented in Table 8 and Table 9. For Example 2, the comparsion showed that, the results are approximatly the same with the exact solution and among the four methods when the degree of each method is equal to 6. However, for Example 3 the comparsion showed that, the other methods are more accurte than the Boubaker polynomials method. Since, if we notice that the numerical solutions which were obtaned from the other methods are approximatly the same with the exact solution when $n = 6$ except for Boubaker polynomials method its still needs degree more than 6 to converged to the exact solution.

Table 8. Comparison of the numerical results for Example 2.

z	Exact Sol.	Boubaker Polynomial (n=6)	Boubaker Polynomial L.S.E	Power Series [9] (n=6)	Power Series [9] Error	Chebyshev Polynomial [9] (n=6)	Chebyshev Polynomial [9] Error	Legender's Polynomial[9] (n=6)	Legender's Polynomial[9] Error
0.0	2.00000	2.00000	0.000000	2.00000	0.00000	2.00016	1.000E-4	2.00251	2.510E-3
0.1	2.57000	2.57000	0.000000	2.55934	1.066E-3	2.57014	1.400E-4	2.57113	1.130E-3
0.2	3.08000	3.08001	0.000000	3.07856	1.440E-3	3.08056	5.600E-4	3.08164	1.640E-3
0.3	3.53000	3.53004	0.000000	3.55621	2.621E-4	3.53027	2.700E-4	3.54712	1.712E-3
0.4	3.92000	3.92007	0.000000	3.94830	2.830E-3	3.91945	5.500E-4	3.91867	1.400E-3
0.5	4.25000	4.25012	0.000000	4.28391	3.391E-3	4.24932	6.800E-4	4.24801	1.990E-3
0.6	4.52000	4.52018	0.000000	4.54167	2.167E-3	4.51814	1.860E-3	4.51772	2.280E-3
0.7	4.73000	4.73026	0.000000	4.74893	1.893E-3	4.73004	4.000E-5	4.73105	1.050E-3
0.8	4.88000	4.88035	0.000000	4.89642	1.642E-3	4.88151	1.510E-3	4.88240	2.400E-3
0.9	4.97000	4.97046	0.000000	4.98341	1.341E-3	4.97793	7.930E-3	4.97806	8.060E-3
1.0	5.00000	5.00059	0.000000	4.99672	1.280E-3	4.99996	4.000E-5	4.99507	1.930E-3

Table 9. Comparison of the numerical results for Example 3.

z	Exact Sol.	Boubaker Polynomial (n=6)	Boubaker Polynomial L.S.E	Power Series [9] (n=6)	Power Series [9] Error	Chebyshev Polynomial [9] (n=6)	Chebyshev Polynomial [9] Error	Legender's Polynomial[9] (n=6)	Legender's Polynomial[9] Error
0.0	1.000000000	1.000000	0.000000	1.000000000	0.000000	1.000000000	0.000000	1.00000000	0.000000
0.1	1.099998477	1.094816	0.000000	1.099816320	1.821E-4	1.099981301	1.718E-5	1.099871453	0.270E-3
0.2	1.199939080	1.179272	0.000000	1.198634511	1.305E-3	1.199947062	7.982E-6	1.199756321	1.828E-4
0.3	1.299986292	1.253391	0.000003	1.299417352	5.689E-4	1.299941073	4.522E-5	1.299861415	1.249E-4
0.4	1.399975631	1.317243	0.000014	1.399094514	8.811E-4	1.399949310	2.632E-5	1.399715631	2.600E-4
0.5	1.499945169	1.370959	0.000043	1.497183216	2.779E-3	1.499960178	1.745E-6	1.499783118	1.788E-4
0.6	1.599945169	1.414760	0.000111	1.587160052	2.785E-3	1.599945831	4.128E-6	1.598345142	1.600E-3
0.7	1.699925370	1.448980	0.000251	1.693861451	6.064E-3	1.699951006	2.564E-5	1.699453417	1.472E-3
0.8	1.799902524	1.474091	0.000511	1.798885324	1.017E-3	1.799960145	5.762E-5	1.799368164	5.344E-4
0.9	1.899876632	1.490730	0.000953	1.889956321	9.920E-3	1.899916315	3.968E-5	1.898964154	9.125E-4
1.0	1.999847695	1.499729	0.001646	1.998794562	1.053E-3	1.999970382	1.227E-4	1.995300161	4.550E-3

COMPARISON BETWEEN TOUCHARD POLYNOMIALS AND POWER SERIES, CHEBYSHEV POLYNOMIALS AND LEGENDER'S POLYNOMIALS

In this section, the comparsion for Example 3 has been presented in Table 10. The comparsion showed that, the Touchard polynomials method converged faster then the other methods. Therefore, these numerical methods are good to estimate the solution of Example 3.

Table 10. Comparison of the numerical results for Example 3.

z	Exact Sol.	Touchard Polynomial [18] (n=5)	Error	Power Series [9] (n=6)	Error	Chebyshev Polynomial [9] (n=6)	Error	Legender's Polynomial [9] (n=6)	Error
0.0	1.000000000	1.000000	0.000000	1.000000000	0.000000	1.000000000	0.000000	1.00000000	0.000000
0.1	1.099998477	1.095000	0.000000	1.099816320	1.821E-4	1.099981301	1.718E-5	1.099871453	0.270E-3
0.2	1.199939080	1.180032	0.000000	1.198634511	1.305E-3	1.199947062	7.982E-6	1.199756321	1.828E-4
0.3	1.299986292	1.255206	0.000000	1.299417352	5.689E-4	1.299941073	4.522E-5	1.299861415	1.249E-4
0.4	1.399975631	1.320738	0.000000	1.399094514	8.811E-4	1.399949310	2.632E-5	1.399715631	2.600E-4
0.5	1.499945169	1.376958	0.000000	1.497183216	2.779E-3	1.499960178	1.745E-6	1.499783118	1.788E-4
0.6	1.599945169	1.424330	0.000001	1.587160052	2.785E-3	1.599945831	4.128E-6	1.598345142	1.600E-3
0.7	1.699925370	1.463466	0.000001	1.693861451	6.064E-3	1.699951006	2.564E-5	1.699453417	1.472E-3
0.8	1.799902524	1.495142	0.000002	1.798885324	1.017E-3	1.799960145	5.762E-5	1.799368164	5.344E-4
0.9	1.899876632	1.520308	0.000001	1.889956321	9.920E-3	1.899916315	3.968E-5	1.898964154	9.125E-4
1.0	1.999847695	1.540112	0.000000	1.998794562	1.053E-3	1.999970382	1.227E-4	1.995300161	4.550E-3

CONCLUSION

As a conclusion, VFIDE2K are required approximate solutions. For this purpose, the numerical methods can be used to obtain approximate solutions for such types of equations. Furthrmore, comparsions among several numerical methods have been studied. The comparsion showed that these numerical methods are reliable numerical techniques for the solution of such types of equations and they can be calculated easily and gives a good results. Moreover, these numerical techniques can be estimate the solution of such equations. For future work, we suggested the study of the comparison among several numerical methods for solving non-linear Volterra-Fredholm integro-differential equations of the second kind.

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