DATA ANALYTICS AND APPLIED MATHEMATICS (DAAM)
e-ISSN: 2773-4854
VOL. 2, ISSUE 1, 020 – 027
DOI: https://doi.org/10.15282/daam.v2i1.5902

ORIGINAL ARTICLE

Some properties of compatible action graph

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ABSTRACT – In this paper, the compatible action graph for the finite cyclic groups of *p*-power order has been considered. The purpose of this study is to introduce some properties of the compatible action graph for finite *p*-groups.

ARTICLE HISTORY
Received: 17/01/2021
Revised: 28/06/2021
Accepted: 29/06/2021

KEYWORDS
Graph theory
Number theory
Cyclic groups
Compatible actions
Nonabelian tensor product

INTRODUCTION

Nowadays, there are many researchers which considered the theoretical relationship between group theory and graph theory. Based on this relationship, several graphs have been established in different ways according to the algebraic structure of the semi group, the group or the ring by using the properties of graph. For example, the directed power graph for the semi group has been defined in [1], and some results on the power graphs of the finite groups are investigated in [2]. Furthermore, the non-commuting graph of rings which is denoted by Γ_R was defined in [3], and two vertices x and y are adjacent whenever $xy \neq yx$. While, the generalized of the conjugate graph $\Gamma_{(G,n)^c}$ has been introduced in [4]. This graph whose vertices are all the non-central subsets of G with n elements and two distinct vertices x and y are adjacent if $x = y^g$ for some $g \in G$. Then, some properties of this graph have been studied such as the chromatic and the independence numbers.

Some of the researchers which studied the graph for the subgroup of a group, such as the coprime graph $P(G)$ of the subgroups of the group G has been defined in [5]. On the other hand, the nilpotent conjugacy class graph of the group G has been provided in [6]. A paper by [7] discussed some properties of the coprime graph Γ_G and classified all finite groups whose coprime graphs are projective. Moreover, the concept of the non-abelian tensor product for two groups which acts on each other provided the actions satisfying the compatibility conditions has been defined in [8]. Based on this concept, the compatible action graph for the nonabelian tensor product for finite cyclic groups of p-power order where p is an odd prime has been introduced in [9]. This graph whose vertices set is the set of $Aut(G)$ and $Aut(H)$ with two distinct vertices ρ and ρ' are adjacent if they are compatible on each other where $\rho \in Aut(G)$ and $\rho' \in Aut(H)$. In this paper, some properties of the compatible action graph for finite p-groups have been studied into two cases which are, where p is an odd prime and $p = 2$.

This paper is organized as follows: Some preliminary results on the compatible actions and compatible action graph with some fundamental concepts of graph theory that are needed in this research are presented in section two. While, the main results of this paper where p is an odd prime are given in section three, and where $p = 2$ are presented in section four. Lastly, the conclusions of this paper are stated in section five.

PRELIMINARY RESULTS

Some definitions and previous results on the compatible actions and compatible action graph with some fundamental concepts on the graph theory are presented in this section. We start with the definition of compatible actions which is given as follows.

Definition 2.1 [8]
Let G and H be the groups which act on each other and each of which act on itself by conjugation. These mutual actions are said to be compatible if $({}^{g}h)_{g'} = {}^{g}({}^{h}({}^{g^{-1}}g'))$ and $({}^{h}g)_{h'} = {}^{h}({}^{g}({}^{h^{-1}}h'))$ for all $g, g' \in G$ and $h, h' \in H$.

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Next, the definition of the compatible action graph for the two finite cyclic groups of the p-power order where p is an odd prime and $p=2$.

Definition 2.2 [9]
Let G and H be two finite cyclic groups of the p-power order where p is an odd prime. Furthermore, let (ρ,ρ') be a pair of the compatible actions for the nonabelian tensor product of $G\otimes H$, where $\rho\in Aut(G)$ and $\rho'\in Aut(H)$. Then, $\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}=\left(V(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}),E(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}})\right)$ is a compatible action graph with the set of vertices $V(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}})$, which is the set of $Aut(G)$ and $Aut(H)$, and the set of edges, $E(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}})$ that connects these vertices, which is the set of all compatible pairs of actions (ρ,ρ'). Furthermore, two vertices ρ and ρ' are adjacent if they are compatible.

Definition 2.3 [10]
Let G and H be two finite cyclic groups of the 2-power order, and (ρ,ρ') be a pair of the compatible actions for the nonabelian tensor product $G\otimes H$ of G and H where $\rho\in Aut(G)$ and $\rho'\in Aut(H)$. Then

$$\Gamma_{G\otimes H}=\big(V(\Gamma_{G\otimes H}),E(\Gamma_{G\otimes H})\big)$$

is a compatible action graph with the set of vertices $V(\Gamma_{G\otimes H})$,which is the set of $Aut(G)$ and $Aut(H)$, and the set of edges $E(\Gamma_{G\otimes H})$ that connects these vertices, which is the set of all compatible pairs of actions (ρ,ρ'). That is

$$V(\Gamma_{G\otimes H})=\begin{cases}Aut(G)\cup Aut(H) & if\ G\neq H,\\ Aut(G) & if\ G=H.\end{cases}$$

Furthermore, two vertices ρ and ρ' are adjacent if they are compatible.

Next, the order of the compatible action graph where p is an odd prime and $p=2$ is given in the following two propositions.

Proposition 2.1 [9]
Let $G=\langle g\rangle\cong C_{p^\alpha}$ and $H=\langle h\rangle\cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha,\beta\geq 3$. Then, the order of the compatible action graph is

i. $\left|V\left(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}\right)\right|=(p-1)(p^{\alpha-1}+p^{\beta-1})$ where $G\neq H$.
ii. $\left|V\left(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}\right)\right|=(p-1)p^{\alpha-1}$ where $G=H$.

Proposition 2.2 [10]
Let $G=\langle g\rangle\cong C_{2^m}$ and $H=\langle h\rangle\cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m\geq 4,n\geq 3$. Then, the order of the compatible action graph $\Gamma_{G\otimes H}$ is

i. $|\Gamma_{G\otimes H}|=2^{m-1}+2^{n-1}$ if $m\neq n$.
ii. $|\Gamma_{G\otimes H}|=2^{m-1}$ if $m=n$.

The out-degree for the given vertex in the compatible action graph where p is an odd prime and $p=2$ is presented in the following two propositions.

Proposition 2.3 [9]
Let $G=\langle g\rangle\cong C_{p^\alpha}$ and $H=\langle h\rangle\cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha,\beta\geq 3$. Furthermore, let $v\in V(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}})$ where $v\in Aut(G)$ and $|v|=p^k$. Then $deg^+(v)$ is one of the following.

i. $(p-1)p^{\beta-1}$ if $k=0$.
ii. $deg^+(v)=(p-1)p^{k-1}+(p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1}$, where $r=\min\{\alpha,\beta\}-k$ if $k=1,2,\dots,\alpha-1$.

Proposition 2.4 [10]
Let $G=\langle g\rangle\cong C_{2^m}$ and $H=\langle h\rangle\cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m\geq 4,n\geq 3$. Furthermore, let $v\in V(\Gamma_{G\otimes H})$ where $v\in Aut(G)$ and $v(g)=g^t$ with $t\equiv(-1)^i\cdot 5^j \pmod{2^m}$ where $i=0,1$, $j=0,1,\dots,2^{m-2}-1$ and $|v|=2^s,s=0,1,\dots,m-2$. Then $deg^+(v)$ is one of the following.

i. 2^{n-1} if $i=j=0$.
ii. 4 if $i=0$ or $i=1$ and $j=2^{m-3}$.
iii. 2^{r-1} if $i=1$ and $j=0$ provided $r=\min\{m+1,n\}$.
iv. 2^s if $i=1,j\neq 0$ and $j\neq 2^{m-3}$.
v. $2^{r'-1}$ if $i=0,j\neq 0$ and $j\neq 2^{m-3}$ provided $r'=\min\{m,n\}$.

The following proposition presented the in-degree for the compatible action graph $\Gamma_{G\otimes H}$ where $p = 2$.

Proposition 2.5 [10]
Let $G = \langle g\rangle \cong C_{2^m}$ and $H = \langle h\rangle \cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m \geq 4, n \geq 3$. Furthermore, let $v \in V(\Gamma_{G\otimes H})$ where $v \in Aut(H)$ and $v(h) = h^t$ with $t \equiv (-1)^i \cdot 5^j \pmod{2^n}$ where $i = 0,1$ and $j = 0,1,\dots,2^{n-2} - 1$ and $|v| = 2^{s'}, s' = 0,1,\dots,n-2$. Then $deg^-(v)$ is one of the following.

i. 2^{m-1} if $i = j = 0$.
ii. 4 if $i = 0$ or $i = 1$ and $j = 2^{n-3}$.
iii. 2^{r-1} if $i = 1$ and $j = 0$ provided $r = \min\{m+1, n\}$.
iv. $2^{s'}$ if $i = 1, j \neq 0$ and $j \neq 2^{n-3}$.
v. $2^{r'-1}$ if $i = 0, j \neq 0$ and $j \neq 2^{n-3}$ provided $r' = \min\{m, n\}$.

Since the compatible action graph is a directed graph, then the out-degree and the in-degree for this graph are the same where p is an odd prime. This property is given in the following corollary.

Corollary 2.1 [9] Let G and H be two finite cyclic groups of p-power order where p is an odd prime and $v \in V(\Gamma^p_{C_{p^\alpha}\otimes C_{p^\beta}})$. Then $deg^+(v) = deg^-(v)$ for $\Gamma^p_{C_{p^\alpha}\otimes C_{p^\beta}}$.

The following corollary shows that the out-degree and the in-degree for the given vertex of compatible action graph $\Gamma_{G\otimes H}$ are equal when $G = H$ and $p = 2$.

Corollary 2.2 [10] Let $G = \langle g\rangle \cong C_{2^m}$ be a finite cyclic group of 2-power order where $m \geq 4$. Then $deg^+(v) = deg^-(v)$ for $\Gamma_{G\otimes G}$.

The connectivity of the compatible action graph has been checked. The following two propositions presented the connectivity of the compatible action graph where p is an odd prime and $p = 2$.

Proposition 2.6 [9] Let $G = \langle g\rangle \cong C_{p^\alpha}$ and $H = \langle h\rangle \cong C_{p^\beta}$ be two finite cyclic groups of the p-power order where p is an odd prime and $\alpha, \beta \geq 3$. Then, $\Gamma^p_{C_{p^\alpha}\otimes C_{p^\beta}}$ is a connected graph.

Proposition 2.7 [10] Let $G = \langle g\rangle \cong C_{2^m}$ and $H = \langle h\rangle \cong C_{2^n}$ be two finite cyclic groups of the 2-power order where $m \geq 4, n \geq 3$. Then, $\Gamma_{C_{2^m}\otimes C_{2^n}}$ is a connected graph.

Next, the characterizations of the Eulerian graph and the Hamiltonian graph are presented in the following two theorems.

Theorem 2.1[11] A connected graph *G* is an Eulerian graph if and only if the degree of each vertex of *G* is even.
Theorem 2.2[11] Let *D* be a strongly connected digraph with *n* vertices. If $deg^+(v) \geq \frac{1}{2n}$ and $deg^-(v) \geq \frac{1}{2n}$ for each vertex *v*, then *D* is Hamiltonian.

The following corollary illustrated that when the graph can be a semi-Eulerian graph.
Corollary 2.3 [11] A connected graph is a semi-Eulerian graph if and only if it has exactly two vertices of odd degree.

Next, some definitions of graph theory that are needed in this research are presented as follows. These definitions can be found in [11] and [12].

The graph *G* is said to be as a planar graph, if it can be drawn in the plan without crossing. In addition, the regular graph is the graph in which each vertex has the same degree. Moreover, the directed graph is called strongly connected graph, if there is a path from *a* to *b* and from *b* to *a* whenever *a* and *b* are vertices in the graph. Finally, the directed graph is said to be a weakly connected graph, if there is a path between every two vertices in the underlying undirected graph.

The following corollary proved that if the group *G* is abelian, then the trivial action is always compatible with any other action.

Corollary 2.4 [13] Let *G* and *H* be groups. Furthermore, let *G* act trivially on *H*. If *G* is abelian, then for any action of *H* on *G*, the mutual actions are compatible.

SOME PROPERTIES OF COMPATIBLE ACTION GRAPH WHERE P IS AN ODD PRIME

In this section, our main results are given where p is an odd prime. Some properties of the compatible action graph for finite cyclic groups of p-power order are presented. We started with the following proposition which presented that the compatible action graph is a planar graph.

Proposition 3.1
Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $v \in V(\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}})$ where v is a trivial action of $Aut(G)$. Then, $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is a planar graph.

Proof: Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $v_1 \in V(\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}})$ where $v_1 \in Aut(G)$ and $|v_1| = 1$, then by Proposition 2.3 (i) $deg^+(v_1) = (p-1)p^{\beta-1}$ without any crossing. Similarly, if we let $v_2 \in V(\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}})$ where v_2 is the trivial action of $Aut(H)$, then by Corollary 2.4, v_2 is compatible with every $v \in Aut(G)$ without any crossing. Thus, $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is a planar graph. ■

The compatible action graph is an Eulerian graph. This result is presented in the following proposition.

Proposition 3.2
Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha, \beta \geq 3$. Then, $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is an Eulerian graph.

Proof: By Proposition 2.6, the compatible action graph $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is a connected graph. Since the compatible action graph $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is considered as a directed graph, then each vertex $v \in V(\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}})$ has an out-degree and an in-degree. But from Corollary 2.1, $deg^+(v) = deg^-(v)$ for the compatible action graph $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$. Thus, we have two cases which are as follows:

Case I: If $|v| = 1$, then by Proposition 2.3(i), $deg^+(v) = (p-1)p^{\beta-1}$ which is an even number for any p is an odd prime.

Case II: If $|v| = p^k$ where $k = 1,2,\dots,\alpha-1$, then by Proposition 2.3 (ii), $deg^+(v) = (p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{\min\{\alpha,\beta\}-k}(p-1)p^{i-1}$ which is an even number for any p is an odd prime.
Thus, from these two cases we have concluded that the compatibe action graph $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is an Eulerian graph. ■

The following corollary shows that the compatible action graph is not a semi Eulerian graph.

Corollary 3.1
Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha, \beta \geq 3$. Then, the compatible action graph $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is not a semi Eulerian graph.

Proof: By Proposition 2.6, $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is a connected graph. Furthermore, let v_1 and v_2 be any two vertices of $V(\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}})$ such that $|v_1| = 1$ and $|v_2| = p^k$ where $k = 1,2,\dots,\alpha-1$ with $v_1, v_2 \in Aut(G)$. Then by Proposition 2.3 (i) and (ii), $deg^+(v_1) = (p-1)p^{\beta-1}$ and $deg^+(v_2) = (p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{\min\{\alpha,\beta\}-k}(p-1)p^{i-1}$. But each of $deg^+(v_1)$ and $deg^+(v_2)$ are even numbers for any p is an odd prime. Thus, $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is not a semi Eulerian graph. ■

The compatible action graph is not a regular graph as given in the following proposition.

Proposition 3.3
Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha, \beta \geq 3$. Then, $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is not a regular graph.

Proof: Assume that the compatible action graph $\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}}$ is a regular graph. Then, according to the definition of a regular graph, any two vertices v_1 and v_2 of $V(\Gamma^p_{C_{p^\alpha} \otimes C_{p^\beta}})$ must have the same degree. Without loss of generality, let $v_1, v_2 \in Aut(G)$ such that $|v_1| = 1$ and $|v_2| = p^k$ where $k = 1,2,\dots,\alpha-1$. Then by Proposition 2.3(i) and (ii), we have $deg^+(v_1) = (p-1)p^{\beta-1}$ and $deg^+(v_2) = (p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{\min\{\alpha,\beta\}-k}(p-1)p^{i-1}$. But $(p-1)p^{\beta-1} \neq$

$(p-1)p^{k-1}+(p-1)p^{k-1}\sum_{i=1}^{\min\{\alpha,\beta\}-k}(p-1)p^{i-1}$ for any p is an odd prime which is a contradiction with our assumption. Therefore, the compatible action graph $\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}$ is not a regular graph. ■

Next, the following proposition proved that the compatible action graph is a strongly connected graph.

Proposition 3.4
Let $G=\langle g\rangle\cong C_{p^\alpha}$ and $H=\langle h\rangle\cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha,\beta\geq 3$. Then, $\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}$ is a strongly connected graph.
Proof: It follows from Proposition 2.3 (i) and Corollary 2.4. ■

The property that the graph is Hamiltonian has been studied. The compatible action graph is a Hamiltonian graph as given in the following proposition.

Proposition 3.5
Let $G=\langle g\rangle\cong C_{p^\alpha}$ and $H=\langle h\rangle\cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha,\beta\geq 3$. Then, $\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}$ is a Hamiltonian graph.
Proof: By Definition 2.2, the compatible action graph is a directed graph, and by Proposition 3.4, the compatible action graph is strongly connected graph. Furthermore, by Corollary 2.1, $deg^{+}(v)=deg^{-}(v)$ for any vertex of the compatible action graph. Thus, we have two cases according to Proposition 2.3, which are as follows:

Case I: Let $v_1\in V(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}})$ such that $|v_1|=1$, then by Proposition 2.3 (i) $deg^{+}(v_1)=(p-1)p^{\beta-1}$, and by Propositions 2.1(i) $\left|V\left(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}\right)\right|=(p-1)(p^{\alpha-1}+p^{\beta-1})$ where $G\neq H$. Now, By Theorem 2.2, $deg^{+}(v_1)=(p-1)p^{\beta-1}\geq\frac{1}{2n}=\frac{1}{2((p-1)(p^{\alpha-1}+p^{\beta-1}))}$ where p is any odd prime.

Case II: Let $v_2\in V(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}})$ such that $|v_2|=p^k$ where $k=1,2,\dots,\alpha-1$. Then by Proposition 2.3(ii), we have $deg^{+}(v_2)=(p-1)p^{k-1}+(p-1)p^{k-1}\sum_{i=1}^{\min\{\alpha,\beta\}-k}(p-1)p^{i-1}$ and by Proposition 2.1(ii), $\left|V\left(\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}\right)\right|=(p-1)p^{\alpha-1}$ where $G=H$. Now, by Theorem 2.2, $deg^{+}(v_2)=(p-1)p^{k-1}+(p-1)p^{k-1}\sum_{i=1}^{\min\{\alpha,\beta\}-k}(p-1)p^{i-1}\geq\frac{1}{2n}=\frac{1}{2((p-1)(p^{\alpha-1}))}$ for any p an odd prime and for all $k=1,2,\dots,\alpha-1$. Thus, from the above two cases, we concluded that the compatible action graph is a Hamiltonian graph. ■

The following proposition shows that the compatible action graph is a weakly connected.

Proposition 3.6
Let $G=\langle g\rangle\cong C_{p^\alpha}$ and $H=\langle h\rangle\cong C_{p^\beta}$ be two finite cyclic groups of p-power order where p is an odd prime and $\alpha,\beta\geq 3$. Then, $\Gamma^{p}_{C_{p^\alpha}\otimes C_{p^\beta}}$ is a weakly connected graph.
Proof: It follows from Proposition 2.3 (i) and Corollary 2.4. ■

SOME PROPERTIES OF COMPATIBLE ACTION GRAPH WHERE P = 2

In this section, the results are presented for such types of groups where $p=2$. Thus, we start with the result that the compatible action graph $\Gamma_{G\otimes H}$ is a planar graph which is given as follows.

Proposition 4.1
Let $G=\langle g\rangle\cong C_{2^m}$ and $H=\langle h\rangle\cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m\geq 4, n\geq 3$. Furthermore, let $v\in V(\Gamma_{G\otimes H})$ where v is a trivial action of $Aut(G)$. Then, $\Gamma_{G\otimes H}$ is a planar graph.

Proof: Let $G=\langle g\rangle\cong C_{2^m}$ and $H=\langle h\rangle\cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m\geq 4, n\geq 3$. Furthermore, let $v_1\in V(\Gamma_{G\otimes H})$ where $v_1\in Aut(G)$ and v_1 be the trivial action, then by Proposition 2.4(i) $deg^{+}(v_1)=2^{n-1}$. Since $|Aut(H)|=2^{n-1}$ then by Corollary 2.4 v_1 is compatible with every $v\in Aut(H)$ without crossing. Similarly, if we have $v_2\in V(\Gamma_{G\otimes H})$ such that v_2 is the trivial action of $Aut(H)$, then by Proposition 2.5(i) $deg^{-}(v_2)=2^{m-1}$ since $|Aut(G)|=2^{m-1}$. Thus, by Corollary 2.4, v_2 is compatible with every $v\in Aut(G)$ without crossing. Therefore, $\Gamma_{G\otimes H}$ is a planar graph. ■

The compatible action graph $\Gamma_{G\otimes H}$ is an Eulerian graph. This result is presented in the following proposition.

Proposition 4.2
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m \geq 4, n \geq 3$. Then, $\Gamma_{G\otimes H}$ is an Eulerian graph.

Proof: By Proposition 2.7, the compatible action graph $\Gamma_{G\otimes H}$ is a connected graph. Since the action of the group G on the group H is the mapping $\Phi: G \to Aut(H)$, then the compatible action graph is a directed graph. Thus, for each vertex $v \in V(\Gamma_{G\otimes H})$ has an out-degree and an in-degree. Therefore, according to Proposition 2.4, each vertex $v \in V(\Gamma_{G\otimes H})$ has an out-degree in five cases which are as follows:

i. If $i = j = 0$ then $deg^+(v) = 2^{n-1}$ which is an even number.
ii. If $i = 0$ or $i = 1$ and $j = 2^{m-3}$ then $deg^+(v) = 4$ which is an even number.
iii. If $i = 1$ and $j = 0$ then $deg^+(v) = 2^{r-1}$ which is an even number provided $r = \min\{m+1, n\}$.
iv. If $i = 1, j \neq 0$ and $j \neq 2^{m-3}$ then $deg^+(v) = 2^s$ which is an even number.
v. If $i = 0, j \neq 0$ and $j \neq 2^{m-3}$ then $deg^+(v) = 2^{r'-1}$ which is an even number provided $r' = \min\{m, n\}$.

Thus, for each vertex $v \in V(\Gamma_{G\otimes H})$ has an even number as an out-degree. Next, by Proposition 2.5, each vertex $v \in V(\Gamma_{G\otimes H})$ has an in-degree in the following five cases which are as follows.

i. If $i = j = 0$ then $deg^-(v) = 2^{m-1}$ which is an even number.
ii. If $i = 0$ or $i = 1$ and $j = 2^{n-3}$ then $deg^-(v) = 4$ which is an even number.
iii. If $i = 1$ and $j = 0$ then $deg^-(v) = 2^{r-1}$ which is an even number provided $r = \min\{m+1, n\}$.
iv. If $i = 1, j \neq 0$ and $j \neq 2^{n-3}$ then $deg^-(v) = 2^{s'}$ which is an even number.
v. If $i = 0, j \neq 0$ and $j \neq 2^{n-3}$ then $deg^-(v) = 2^{r'-1}$ which is an even number provided $r' = \min\{m, n\}$.

Thus, for each vertex $v \in V(\Gamma_{G\otimes H})$ has an even number as an in-degree. Hence, $\Gamma_{G\otimes H}$ is an Eulerian graph. ■

Corollary 4.1 Let $G = H \cong C_{2^m}$ be a finite cyclic group of 2-power order where $m \geq 4$. Then, $\Gamma_{G\otimes H}$ is an Eulerian graph.

Proof: It follows from Corollary 2.2, Proposition 2.7, Proposition 2.4 and Theorem 2.1. ■

The following corollary shows that the compatible action graph is not a semi Eulerian graph.

Corollary 4.2
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m \geq 4, n \geq 3$. Then, the compatible action graph $\Gamma_{G\otimes H}$ is not a semi Eulerian graph.

Proof: By Proposition 2.7, the compatible action graph $\Gamma_{G\otimes H}$ is a connect graph. Assume that the compatible action graph $\Gamma_{G\otimes H}$ is a semi Eulerian Graph, then by Corollary 2.3, there exist two vertices $v_1, v_2 \in V(\Gamma_{G\otimes H})$ both of them having an odd degree. Now, let $v_1, v_2 \in V(\Gamma_{G\otimes H})$ where $v_1, v_2 \in Aut(G)$ such that v_1 is the trivial action of $Aut(G)$, and v_2 be a non-tivial action of $Aut(G)$. Then by Proposition 2.4(i),(ii),(iii),(iv) and (v), $deg^+(v_1) = 2^{n-1}$ and $deg^+(v_2)$ is of 2-power order. Therefore, we conclude that v_1 and v_2 are of even degree which is a contradiction with our assumption. Therefore, $\Gamma_{G\otimes H}$ is not a semi Eulerian graph. ■

The following proposition shows that the compatible action graph $\Gamma_{G\otimes H}$ is not a regular graph.

Proposition 4.3
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m \geq 4, n \geq 3$. Then, $\Gamma_{G\otimes H}$ is not a regular graph.

Proof: Assume that, the compatible action graph $\Gamma_{G\otimes H}$ is a regular graph. Then, according to the definition of a regular graph, each vertex has the same degree. Thus, let v_1 and v_2 be any two vertices of $V(\Gamma_{G\otimes H})$, then they must have the same degree. Without loss of generality, let $v_1, v_2 \in Aut(H)$ such that $v_1(h) = h^t$ with $t \equiv (-1)^i \cdot 5^j \pmod{2^n}$ where $i = j = 0$, and $v_2(h) = h^{t'}$ with $t' \equiv (-1)^i \cdot 5^j \pmod{2^n}$ where $i = 1, j \neq 0$ and $j \neq 2^{n-3}$. Then by Proposition 2.5(i) and (iv) $deg^-(v_1) = 2^{m-1} \neq deg^-(v_2) = 2^{s'}$ which is a contradiction with our assumption. Thus, $\Gamma_{G\otimes H}$ is not a regular graph. ■

Next, the compatible action graph $\Gamma_{G\otimes H}$ is a strongly connected graph as presented in the following proposition.

Proposition 4.4
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m \geq 4, n \geq 3$. Then, $\Gamma_{G \otimes H}$ is strongly connected.

Proof: It follows from Proposition 2.4(i), Proposition 2.5(i) and Corollary 2.4. ■

The following proposition proved that the compatible action graph is $\Gamma_{G \otimes H}$ is a Hamiltonian graph.

Proposition 4.5
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m \geq 4, n \geq 3$. Then, $\Gamma_{G \otimes H}$ is a Hamiltonian graph.

Proof: Since the action of the group G on the group H is the mapping $\varphi: G \to Aut(H)$, then the compatible action graph is a directed graph. By Proposition 4.4, the compatible action graph is a strongly connected graph. Furthermore, by Proposition 2.2, $|\Gamma_{G \otimes H}| = 2^{m-1} + 2^{n-1}$ when $m \neq n$ and $|\Gamma_{G \otimes H}| = 2^{m-1}$ when $m = n$. Thus, we have two cases according to the order of the compatible action graph which are as follows:

Case I: Let $G = H$, then by Corollary 2.2, $deg^+(v) = deg^-(v)$. Thus, by Proposition 2.4 and Theorem 2.2 we have:

i. $deg^+(v) = deg^-(v) = 2^{m-1} \geq \frac{1}{2(2^{m-1})} = \frac{1}{2^m}$.
ii. $deg^+(v) = deg^-(v) = 4 \geq \frac{1}{2(2^{m-1})} = \frac{1}{2^m}$.
iii. $deg^+(v) = deg^-(v) = 2^{r-1} \geq \frac{1}{2(2^{m-1})} = \frac{1}{2^m}$.
iv. $deg^+(v) = deg^-(v) = 2^s \geq \frac{1}{2(2^{m-1})} = \frac{1}{2^m}$.

Case II: Let $G \neq H$, then by Proposition 2.4, Proposition 2.5 and Theorem 2.2 we have

i. $deg^+(v) = 2^{n-1} \geq \frac{1}{2(2^{m-1}+2^{n-1})} = \frac{1}{2^m+2^n}$, where $i = j = 0$.
ii. $deg^+(v) = 4 \geq \frac{1}{2(2^{m-1}+2^{n-1})} = \frac{1}{2^m+2^n}$, where $i = 0$ or $i = 1$ and $j = 2^{m-3}$.
iii. $deg^+(v) = 2^{r-1} \geq \frac{1}{2(2^{m-1}+2^{n-1})} = \frac{1}{2^m+2^n}$, where $i = 1, j = 0$ and $r = \min\{m+1, n\}$.
iv. $deg^+(v) = 2^s \geq \frac{1}{2(2^{m-1}+2^{n-1})} = \frac{1}{2^m+2^n}$, where $i = 1, j \neq 0, j \neq 2^{m-3}$.
v. $deg^+(v) = 2^{r'-1} \geq \frac{1}{2(2^{m-1}+2^{n-1})} = \frac{1}{2^m+2^n}$, $i = 0, j \neq 0, j \neq 2^{m-3}$ and $r' = \min\{m, n\}$.
vi. $deg^-(v) = 2^{m-1} \geq \frac{1}{2(2^{m-1}+2^{n-1})} = \frac{1}{2^m+2^n}$, where $i = j = 0$.
vii. $deg^-(v) = 4 \geq \frac{1}{2(2^{m-1}+2^{n-1})}$, where $i = 0$ or $i = 1$ and $j = 2^{n-3}$.
viii. $deg^-(v) = 2^{r-1} \geq \frac{1}{2(2^{m-1}+2^{n-1})}$, where $i = 1, j = 0$ and $r = \min\{m+1, n\}$.
ix. $deg^-(v) = 2^s \geq \frac{1}{2(2^{m-1}+2^{n-1})}$, where $i = 1, j \neq 0$ and $j = 2^{n-3}$.
x. $deg^-(v) = 2^{r'-1} \geq \frac{1}{2(2^{m-1}+2^{n-1})}$, where $i = 0, j \neq 0, j \neq 2^{n-3}$ and $r = \min\{m, n\}$.

Thus, from the above two cases, and by Theorem 2.2, we concluded that the compatible action graph is a Hamiltonian graph. ■

Finally, the compatible action graph $\Gamma_{G \otimes H}$ is a weakly connected graph as given in the following proposition.

Proposition 4.6
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be two finite cyclic groups of 2-power order where $m \geq 4, n \geq 3$. Then, $\Gamma_{G \otimes H}$ is a weakly connected graph.

Proof: It follows from Proposition 2.4(i), Proposition 2.5(i) and Corollary 2.4. ■

CONCLUSION

In this paper, some properties of the compatible action graph for the nonabelian tensor product for finite cyclic groups of p-power order have been studied. Furthermore, the obtained results showed that the compatible action graph is a planar graph, Eulerian graph, strongly connected graph, Hamiltonian graph and weakly connected graph but it is not a regular graph.

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