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ORIGINAL ARTICLE

The compatible action graphs for finite cyclic 2-groups

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ABSTRACT – A graph namely the compatible action graph is introduced by extending the results on the compatible pairs of actions for cyclic 2-groups. Some properties of compatible action graphs are presented. In addition, the special types of compatible action graphs are given.

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INTRODUCTION

In [1] defined the nonabelian tensor product as a pair of groups, G and H which acts on each other provided the actions satisfy the compatibility conditions:

$$^{\left({}^{g}h\right)}g' = {}^{g}\left({}^{h}\left({}^{g^{-1}}g'\right)\right) \text{ and } ^{\left({}^{h}g\right)}h' = {}^{h}\left({}^{g}\left({}^{h^{-1}}h'\right)\right),$$

for all $g, g' \in G$ and $h, h' \in H$. If G and H are groups that act compatibly with each other, then $G \otimes H$ is group generated by $g \otimes h$ with these two relations

$$gg' \otimes h = \left({}^{g}g' \otimes {}^{g}h\right)\left(g \otimes h\right) \text{ and } g \otimes hh' = \left(g \otimes h\right)\left({}^{h}g \otimes {}^{h}h'\right)$$

for all $g, g' \in G$ and $h, h' \in H$.

There are some researcher study on compatible action of cyclic groups, for example in [2] and in [3] studied on compatibility condition of finite cyclic groups of p-power order. Then, in [4] studied on the number of compatible pair of action for cyclic groups of 2-power order. Next, in [5] studied on number of compatible pair of actions for finite cyclic groups of 3-power order. In [6] continues the studies on compatible pair of nontrivial action for finite cyclic 2-groups. After that, in [7] studies on compatible number of compatible pair of action for finite cyclic groups of p-power order.

Most of the researchers interested in the study of algebraic structure especially in the properties of graphs. They investigate the interplay between group theory and graph theory. In [8] studied on the properties of the non-commuting graph of G denoted by Γ_G where G is a nonabelian group and $Z(G)$ is the center of G. Later, in [9] studied on the non-commuting graph for two groups G and H where G is the nonabelian finite groups and H is the finite nonabelian simple Lie group. While in [10] presented the conditions on the edge and vertices of a non-commuting graph. In [11] defined the non-coprime graph associated and present properties of non-coprime graph. In [12] presented on subgraph of compatible action graph for finite cyclic groups of p-power order but for the p only odd prime only.

The motivation of this paper that is to investigate the compatible action graph for subgroups of cyclic 2-groups is that, usually, the compatible pairs of actions for the subgroup H from the group G should exist in the group G, but it is not necessary. The following example is given to show that there are compatible pairs of actions which exist in the subgroup $C_{2^4} \otimes C_{2^4}$ but not in the group $C_{2^5} \otimes C_{2^5}$.

Example 1
Let $G = \langle g \rangle \cong C_{2^5}$ and $H = \langle h \rangle \cong C_{2^5}$ be cyclic 2-groups. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ be two actions such that $\sigma(g) = g^5$ and $\sigma'(h) = h^5$ or equivalently ${}^{g}h = h^5$ and ${}^{h}g = g^5$ for all $g, g' \in G$ and $h, h' \in H$.

Then, for the first compatibility condition,

$$\begin{aligned} {}^{{}^{h}g}h &= {}^{g^5}h \\ &= h^{5^5} \\ &= h^{21} \text{ since } 5^5 \equiv 21 \left(\operatorname{mod} 2^5\right) \\ &\neq {}^{g}h. \end{aligned}$$

Thus, (σ,σ') is not compatible in $C_{2^5} \otimes C_{2^5}$. Now, let $G=\langle g\rangle \cong C_{2^4}$ and $H=\langle h\rangle \cong C_{2^4}$ be cyclic 2-groups. Furthermore, let $\sigma(g)=g^5$ and $\sigma'(h)=h^5$ or equivalently ${}^{g}h=h^5$ and ${}^{h}g=g^5$. Then, for the first compatibility condition,

$$\begin{aligned} {}^{{}^{h}g}h &= {}^{g^5}h \\ &= h^{5^5} \\ &= h^{5} \quad \text{since } 5^5 \equiv 5 \left(\operatorname{mod} 2^4\right) \\ &= {}^{g}h. \end{aligned}$$

Similarly, the second compatibility conditions is also satisfied. Thus, (σ,σ') is compatible in $C_{2^4} \otimes C_{2^4}$. Thus, there are compatible pairs of actions which exist in the subgroup $C_{2^4} \otimes C_{2^4}$ but not in the group $C_{2^5} \otimes C_{2^5}$.

Example 2 illustrated the idea and the intersection between the group and the subgroup.

Example 2

Let $G \cong C_{2^5}$ be a cyclic 2-groups. Furthermore, let $H \cong C_{2^4}$ be a subgroup of *G*. For the nonabelian tensor product of the subgroup, $C_{2^4} \otimes C_{2^4}$, there are three compatible pairs of actions (σ,σ'), which are $(g^5,h^5), (g^7,h^7)$ and (g^{15},h^{15}) are compatible in $C_{2^4} \otimes C_{2^4}$ but not in $C_{2^5} \otimes C_{2^5}$.

However, there are some compatible pairs of actions (σ,σ') such as $(g^5,h^9),(g^9,h^5),(g^9,h^9),(g^9,h^{13})$ and (g^{13},h^9) are compatible in $C_{2^4} \otimes C_{2^4}$ and also $C_{2^5} \otimes C_{2^5}$. Thus, all of the compatible pairs of actions represented as intersection between both nonabelian tensor product.

From the literature, there are many researchers investigate the specific graph on groups and determine the graph properties. Therefore, our interest to introduce the associated compatible action graph of the finite cyclic 2-groups and their properties.

PRELIMINARY RESULTS

Some definitions and previous results on compatible conditions that will be used to determine the properties of compatible action graph are stated. It starts with the definition of an action of group *G* on group *H* and compatible pair of action between two groups, which is as follows.

Definition 1: Action [2]

Let *G* and *H* be cyclic groups. An action of *G* on *H* is a mapping $\Phi : G \rightarrow \operatorname{Aut}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)(\Phi(g')(h)),$$

for all $g, g' \in G$ and $h \in H$.

Definition 2: Compatible Action [1]

Let *G* and *H* be groups which act on each other. These mutual actions are said to be compatible with each other and with the actions of *G* and *H* on themselves by conjugation if

$$^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } {}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h')),$$

for all $g, g' \in G$ and $h, h' \in H$.

According to the presentation of the automorphism group of a cyclic 2-group, Mohamad (2012) given the necessary and sufficient conditions for a pair of actions that act compatibly with each other with specific order. If one of the actions has order two, then the necessary and sufficient conditions for the other actions to act compatibly with each other are given in the following theorem.

Theorem 1 [3]
Let $G=\langle x\rangle \cong C_{2^m}$ and $H=\langle y\rangle \cong C_{2^n}$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma|=2$ and $\sigma' \in \mathrm{Aut}(H)$, where $m\geq 2, n\geq 3$.

i. If $\sigma(x)=x^t$ with $t\equiv -1\left(\bmod\ 2^m\right)$ or $t\equiv 2^{m-1}-1\left(\bmod\ 2^m\right)$, then (σ,σ') is a compatible pair if and only if σ' is the trivial automorphism or $|\sigma'|=2$.
ii. If $\sigma(x)=x^t$ with $t\equiv 2^{m-1}+1\left(\bmod\ 2^m\right)$, then (σ,σ') is a compatible pair if and only if $|\sigma'|\leq 2^{s'}$ with $s'\leq m-1$. In particular, σ is compatible with all $\sigma'\in \mathrm{Aut}(H)$ provided $n\leq m+1$.

Next, the necessary and sufficient conditions of compatible conditions where one of the actions has an order greater than two are stated in the following theorem.

Theorem 2 [3]
Let $G=\langle x\rangle \cong C_{2^m}$ and $H=\langle y\rangle \cong C_{2^n}$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma|=2^s, s\geq 2$ and $\sigma' \in \mathrm{Aut}(H)$, where $m\geq 4, n\geq 1$.

i. If $\sigma(x)=x^t$ with $t\equiv -5^j\left(\bmod\ 2^m\right)$, then (σ,σ') is a compatible pair if and only if $\sigma'(y)=y^{t'}$ with $t'\equiv 1\left(\bmod\ 2^n\right)$ or $t'\equiv 2^{n-1}+1\left(\bmod\ 2^m\right)$.
ii. If $\sigma(x)=x^t$ with $t\equiv 5^j\left(\bmod\ 2^m\right)$, then (σ,σ') is a compatible pair if and only if $|\sigma'|\leq 2^{m-s}$ provided $n\leq m-s+2$.

Proposition 2.1 gives the compatibility conditions when one of the actions is trivial with *G* and *H* are cyclic groups.

Proposition 1 [3]
Let $C_m=\langle x\rangle$ and $C_n=\langle y\rangle$ be finite cyclic groups and act on each other. If one of the actions is trivial, then any pair of actions of $C_m=\langle x\rangle$ and $C_n=\langle y\rangle$ are compatible.

Next, the number of the compatible pair of actions can be determined by using the necessary and sufficient conditions for two cyclic groups of 2-power order to act compatibly on each other.
First, let one of the actions has an order one, then the number of the compatible pair of actions is given as follows.

Proposition 2 [5]
Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 1, n\geq 1$. If *G* acts trivially on *H*, then the number of compatible pairs of actions is 2^{n-1}.

The number of compatible pairs of actions for cyclic 2-groups when one of the actions has an order two is presented in Proposition 2.3 and Theorem 2.3 respectively.

Proposition 3 [5]
Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 2, n\geq 3$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma|=2$ and $\sigma' \in \mathrm{Aut}(H)$ such that the pair (σ,σ') acts compatibly with each other.

i. If $\sigma(x)=x^t$ with $t\equiv -1\ (\bmod\ 2^m)$ or, then there are eight compatible pairs (σ,σ').
ii. If $\sigma(x)=x^t$ with $t\equiv 2^{m-1}+1\ \left(\bmod\ 2^m\right)$, then there are 2^{r-1} compatible pairs (σ,σ') where $r=\min\{m+1,n\}$.

Theorem 3 [5]
Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 2, n\geq 3$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma|=2$. Then there are $2^{r-1}+8$ compatible pairs of actions (σ,σ') where $\sigma'\in \mathrm{Aut}(H)$ with $r=\min\{m+1,n\}$.

The following Lemmas presented the number of compatible pairs of actions for cyclic 2-groups where one of the actions has an order greater than two given in Lemmas 2.1 and 2.2 respectively.

Lemma 1 [5]
Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 4, n\geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma|=2^s, s\geq 2$ and $\sigma' \in \text{Aut}(H)$. If $\sigma(g)=g^t$ with $t\equiv -5^j \pmod{2^m}$, then there are 2^s compatible pairs (σ,σ').

Lemma 2 [5]
Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 4, n\geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma|=2^s, s\geq 2$ and $\sigma' \in \text{Aut}(H)$ with $|\sigma'|=2^{s'}$. If $\sigma(g)=g^t$ with $t\equiv 5^j \pmod{2^m}$, then there are $2^{r'-1}$ compatible pairs (σ,σ') with $r'=\min\{m,n\}$.

Next, the total number of compatible pairs of actions for two cyclic 2-groups as follows.

Theorem 4 [5]
Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 4, n\geq 3$. Then, there exist

$$(m-3)(2^{r'-1})+2^{r-1}+2^{m-1}+2^{n-1}+4$$

compatible pairs of actions where $r=\min\{m+1,n\}$ and $r'=\min\{m,n\}$.

Definition 3: Order of a Graph [13]
The order of a graph *G* is the number of vertices in *G*. It is denoted by $|G|$. Thus $|G|=|V(G)|$.

THE RESULTS ON PROPERTIES OF COMPATIBLE ACTION GRAPH

The definition of compatible action graph is given as follows.

Definition 4: Compatible Action Graph
Let *G* and *H* be finite cyclic 2-groups and (σ,σ') be a pair of compatible actions for the nonabelian tensor product $G\otimes H$, where $\sigma\in \text{Aut}(G)$ and $\sigma'\in\text{Aut}(H)$. Then,

$$\Gamma_{G\otimes H}=(V(\Gamma_{G\otimes H}), E(\Gamma_{G\otimes H}))$$

is a compatible action graph with the set of the vertices $V(\Gamma_{G\otimes H})$, which is the union of Aut(*G*) and Aut(*H*), and the set of edges, $E(\Gamma_{G\otimes H})$ that connect these vertices which is the set of all compatible pairs of actions (σ,σ'). That is

$$V(\Gamma_{G\otimes H})=\begin{cases}\text{Aut}(G)\cup\text{Aut}(H) & \text{if } G\neq H,\\ \text{Aut}(G) & \text{if } G=H.\end{cases}$$

Furthermore, the vertices σ and σ' are adjacent if they are compatible.

The order of Γ is defined as the cardinality of the vertex set of Γ. Since the vertices of the compatible action graph are elements of Aut(*G*) and Aut(*H*), then the order of the compatible action graph is the number of automorphisms of *G* and *H*. Therefore, $|\Gamma_{G\otimes H}|=|V(\Gamma_{G\otimes H})|$. The order of the compatible action graph is considered for two cases namely $m\neq n$ and $m=n$. Thus, the following proposition presented the order of the compatible action graph for the cyclic 2-groups.

Proposition 4
Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 4, n\geq 3$. Then, the order of the compatible action graph of *G* and *H* is:

i. $\left|\Gamma_{G\otimes H}\right| = 2^{m-1} + 2^{n-1}$ if $m \neq n$.

ii. $\left|\Gamma_{G\otimes H}\right| = 2^{m-1}$ if $m = n$.

Proof Let $G = \langle g\rangle \cong C_{2^m}$ and $H = \langle h\rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. By Definition 3, $\left|\Gamma_{G\otimes H}\right| = \left|V(\Gamma_{G\otimes H})\right|$. Furthermore, by Definition 4, $V(\Gamma_{G\otimes H})$ is the set of Aut(G) and Aut(H). There are two cases which are $m \neq n$ and $m = n$ need to be considered.

i. Let $m \neq n$ and note that $\left|V(\Gamma_{G\otimes H})\right| = \left|\mathrm{Aut}(G)\right| + \left|\mathrm{Aut}(H)\right|$ where $\left|\mathrm{Aut}(G)\right| = 2^{m-1}$ and $\left|\mathrm{Aut}(H)\right| = 2^{n-1}$. Then, $\left|\Gamma_{G\otimes H}\right| = 2^{m-1} + 2^{n-1}$.

ii. Let $m = n$. Without loss of generality, let $|G| = |H| = 2^m$. Since $\left|V(\Gamma_{G\otimes H})\right| = \left|\mathrm{Aut}(G)\right|$ and $\left|\mathrm{Aut}(G)\right| = 2^{m-1}$. Then, $\left|\Gamma_{G\otimes H}\right| = 2^{m-1}$. □

The compatible action graph of cyclic 2-groups is a directed graph since an action of G on H is a mapping $\Phi: G \to$ Aut(H). The compatible action graph may contain a loop and the loop is only present for the case $G = H$. Thus, the loop contributes one to both of the in-degree and the out-degree of the vertex. The compatible action graph may not contain multiple directed edges because the presentation of automorphism are not repeated given by the necessary and sufficient conditions of the compatible on each other for cyclic 2-groups. The following proposition presented the number of the edge for the compatible action graph for the cyclic 2-groups.

Proposition 5
Let $G = \langle g\rangle \cong C_{2^m}$ and $H = \langle h\rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then,

$$\left|E(\Gamma_{G\otimes H})\right| = (m-3)(2^{r'-1}) + 2^{r-1} + 2^{m-1} + 2^{n-1} + 4,$$

where $r = \min\{m+1,\ n\}$ and $r' = \min\{m,\ n\}$.

Proof Let $G = \langle g\rangle \cong C_{2^m}$ and $H = \langle h\rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. By Definition 4, the $E(\Gamma_{G\otimes H})$) is a nonempty set of all pairs of (σ, σ'). Thus, by Theorem 4, $\left|E(\Gamma_{G\otimes H})\right| = (m-3)(2^{r'-1}) + 2^{r-1} + 2^{m-1} + 2^{n-1} + 4$ where $r = \min\{m+1, n\}$ and $r' = \min\{m, n\}$. □

In the terminology of graphs, directed edges reflect the fact that the edges in a directed graph have directions. Thus, (σ, σ') can be defined as the edge of the compatible action graph with the directed edge. The vertex σ is considered as initial vertex and σ' as terminal vertex. In addition, σ is to be adjacent to σ' and σ' is to be adjacent from σ The following proposition presented the out-degree of vertex v for compatible action graph.

Proposition 6
Let $G = \langle g\rangle \cong C_{2^m}$ and $H = \langle h\rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $v \in V(\Gamma_{G\otimes H})$ where $v \in \mathrm{Aut}(G)$ and $v(g) = g^t$ with $t \equiv (-1)^i \cdot 5^j \left(\bmod 2^m\right)$ where $i = 0, 1$ and $j = 0, 1, \ldots, 2^{m-2} - 1$ and $|v| = 2^s$, $s = 0, 1, \ldots, m-2$. Then, $\deg^+(v)$ is exactly one of the following:

i. 2^{n-1} if $i = 0$ and $j = 0$.

ii. 4 if $i = 0$ or $i = 1$ and $j = 2^{m-3}$.

iii. 2^{r-1} if $i = 1$ and $j = 0$ provided $r = \min\{m+1, n\}$.

iv. 2^s if $i = 1$, $j \neq 0$ and $j \neq 2^{m-3}$.

v. $2^{r'-1}$ if $i = 0$, $j \neq 0$ and $j \neq 2^{m-3}$ provided $r' = \min\{m, n\}$.

Proof Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 4, n\geq 3$. Furthermore, let $v\in V(\Gamma_{G\otimes H})$ where $v\in$ Aut(G) and $v(g)=g^t$ with $t\equiv(-1)^i\cdot 5^j \pmod{2^m}$ where $i=0,1$ and $j=0,1,\ldots,2^{m-2}-1$ and $|v|=2^s$, $s=0,1,\ldots,m-2$. There are five cases according to the $\deg^+(v)$.

i. Let $i=0$ and $j=0$, then the action is trivial. By Proposition 2, the action is compatible when the action of H on G is trivial. Thus, $\deg^+(v)=2^{n-1}$.

ii. By Proposition 3(i), there are eight compatible pairs of actions. Let $i=0$ and $j=2^{m-3}$, then there are four compatible pairs of actions. The number of compatible pairs of actions is the same when $i=1$ and $j=2^{m-3}$. Particularly, the compatible pairs of actions are four for $i=0$ and four for $i=1$. Therefore, $\deg^+(v)=4$ where $i=0$ or $i=1$ and $j=2^{m-3}$.

iii. By Proposition 3(ii), there are 2^{r-1} compatible pairs of actions for $t\equiv 2^{m-1}+1 \pmod{2^m}$ provided $r=\min\{m+1,n\}$. Thus, $\deg^+(v)=2^{r-1}$ where $i=1$ and $j=0$.

iv. By Lemma 1, the number of compatible pairs of actions is 2^s. Therefore, $\deg^+(v)=2^s$ when $i=1$, $j\neq 0$ and $j\neq 2^{m-3}$.

v. By Lemma 2, there are $2^{r'-1}$ number of the compatible pairs of actions where $i=0$, $j\neq 0$ and $j\neq 2^{m-3}$ provided $r'=\min\{m,n\}$. Thus, the $\deg^+(v)=2^{r'-1}$. □

The graph with directed edges the in-degree of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The following proposition presented in-degree of vertex v.

Proposition 7

Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 4, n\geq 3$. Furthermore, let $v\in V(\Gamma_{G\otimes H})$ where $v\in \mathrm{Aut}(H)$ and $v(h)=h^t$ with $t\equiv(-1)^i\cdot 5^j \pmod{2^n}$ where $i=0,1$ and $j=0,1,\ldots,2^{n-2}-1$ and $|v|=2^{s'}$, $s'=0,1,\ldots,n-2$. Then, $\deg^-(v)$ is exactly one of the following:

i. 2^{m-1} if $i=0$ and $j=0$.

ii. 4 if $i=0$ or $i=1$ and $j=2^{n-3}$.

iii. 2^{r-1} if $i=1$ and $j=0$ provided $r=\min\{m+1,n\}$.

iv. $2^{s'}$ if $i=1$, $j\neq 0$ and $j\neq 2^{n-3}$.

v. $2^{r'-1}$ if $i=0$, $j\neq 0$ and $j\neq 2^{n-3}$ provided $r'=\min\{m,n\}$.

Proof Let $G=\langle g\rangle \cong C_{2^m}$ and $H=\langle h\rangle \cong C_{2^n}$ be cyclic groups where $m\geq 4, n\geq 3$. Furthermore, let $v\in V(\Gamma_{G\otimes H})$ where $v\in \mathrm{Aut}(H)$ and $v(h)=h^t$ with $t\equiv(-1)^i\cdot 5^j \pmod{2^n}$ where $i=0,1$ and $j=0,1,\ldots,2^{n-2}-1$ and $|v|=2^{s'}$, $s'=0,1,\ldots,n-2$. There are five cases according to $\deg^-(v)$.

i. Let $i=0$ and $j=0$, the action is trivial. By Proposition 2, the action is compatible when the action of G on H is trivial. Thus, $\deg^-(v)=2^{m-1}$.

ii. By Proposition 3(i), there are eight compatible pairs of actions. Let $i=0$ and $j=2^{n-3}$, then there are four compatible pairs of actions. The number of compatible pairs of actions is the same when $i=1$ and $j=2^{n-3}$. Particularly, the compatible pairs of actions are four for $i=0$ and four for $i=1$. Therefore, $\deg^-(v)=4$ where $i=0$ or $i=1$ and $j=2^{n-3}$.

iii. By Proposition 3(ii), there are 2^{r-1} compatible pairs of actions provided $r=\min\{m+1,n\}$. Thus. $\deg^-(v)=2^{r-1}$ where $i=1$ and $j=0$.

iv. By Lemma 1, the number of the compatible pairs of actions is $2^{s'}$. Thus, $\deg^-(v) = 2^{s'}$ where $i = 1, j \neq 0$ and $j \neq 2^{n-3}$.

v. By Lemma 2, given that there are $2^{r'-1}$ number of the compatible pairs of actions where $i = 0, j \neq 0$ and $j \neq 2^{n-3}$ where $r' = \min\{m,n\}$. Thus, $\deg^-(v)2^{r'-1}$. □

Particulary, the following corollary shows that the out-degree of vertex v and the in-degree of vertex v are equal for the compatible action graph when $G = H$.

Corollary 1
Let $G = \langle g \rangle \cong C_{2^m}$ be a cyclic group where $m \geq 4$. Then $\deg^-(v) = \deg^+(v)$ for $\Gamma_{G \otimes G}$.

Proof Let $G = \langle g \rangle \cong C_{2^m}$ be a cyclic group where $m \geq 4$. By Propositions 6 and 7, the $\deg^-(v) = \deg^+(v)$ for any $v = V(\Gamma_{G \otimes G})$. □

TYPES OF COMPATIBLE ACTION GRAPH

First, the connectivity of compatible actions graphs is investigated. A compatible action graph is connected when there is a path between any pair of vertices. The connectivity of the compatible action graph for the cyclic 2-groups is given in the following theorem.

Theorem 5
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then, $\Gamma_{G \otimes H}$ is a connected graph.

Proof Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $v_1 \in V(\Gamma_{G \otimes H})$ with $v_1 \in \text{Aut}(G)$ and $\deg^- v_1 = 2^{n-1}$. Then, v_1 is compatible with every $v_2 \in \text{Aut}(H)$ since $|\text{Aut}(H)| = 2^{n-1}$.

Next, let $v_2 \in V(\Gamma_{G \otimes H})$ with $v_2 \in \text{Aut}(H)$ and $\deg^+(v_2) = 2^{m-1}$. Then, v_2 is compatible with every $v_1 \in \text{Aut}(G)$ since $|\text{Aut}(G)| = 2^{m-1}$. Thus, $\Gamma_{G \otimes H}$ is a connected graph. □

Next, let $G \neq H$. Then, the compatible action graph has the property where the vertex can be partitioned into two disjoints sets namely V_1 and V_2 or equivalently the graph is bipartite. This result is provided in the following theorem.

Theorem 6
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then, $\Gamma_{G \otimes H}$ is a bipartite graph if and only if $m \neq n$.

Proof Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. First, we need to show that if $\Gamma_{G \otimes H}$ is bipartite, then $m \neq n$. By using contradiction method, assume that $\Gamma_{G \otimes H}$ is bipartite and $m = n$ is true. Assume that $m = n$, then $\text{Aut}(C_{2^m}) = \text{Aut}(C_{2^n})$. Thus, there exist a loop which cannot be partitioned into two disjoint sets, which contradicts on the assumption. Therefore, $m \neq n$.

Next, let $m \neq n$. By definition of compatible pairs of actions, if $m \neq n$, then any $v \in \text{Aut}(C_{2^m})$ only compatible with some $v \in \text{Aut}(C_{2^n})$. Thus, clearly it can be partitioned into two disjoint sets $\text{Aut}(C_{2^m})$ and $\text{Aut}(C_{2^n})$ respectively. Therefore, $\Gamma_{G \otimes H}$ is a bipartite graph. □

Next, the complete graph is investigated. The complete graph contains exactly one edge between each pair of vertices. The result shows the compatible action graphs is not the complete graph. The result is given as follows

Theorem 7
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then, $\Gamma_{G \otimes H}$ is not a complete graph.

Proof By Proposition 3.1, the number of vertices for compatible action graphs is $2^{m-1}+2^{n-1}$. If a compatible action graph is a complete graph, the number of edges is $\left[2^{m-1}+2^{n-1}\right]^2$. By Proposition 5, there exist $\left|E(\Gamma_{G\otimes H})\right| = 2^{n-1}+2^{r-1}+4+2^{m-1}+\left(m-3\right)2^{r'-1}$ edges where $r=\min\{m+1,n\}$ and $r'=\min\{m,n\}$. Thus,

$$\left|E(\Gamma_{G\otimes H})\right| = 2^{n-1}+2^{r-1}+4+2^{m-1}+\left(m-3\right)2^{r'-1} \le [2^{m-1}+2^{n-1}]^2.$$

Therefore, the compatible action graph is not a complete graph. □

CONCLUSION

In this paper, the compatible action graph is introduced. Consequently, some properties of the compatible actions graph are stated such that the cardinality of the edge set, order of a compatible action graph, the number of directed edges in the in-degree and out-degree of a vertex. From a results show that the types of compatible action graph are connected, bipartite and not the complete graph.

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