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RESEARCH ARTICLE

# Determining the differential operator boundary value problems on graphs

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**ABSTRACT** - The behaviour of functions on graph structures is studied mathematically in the context of differential operator boundary value issues on graphs. The finding of differential operators for graph boundary value issues is the focus of this abstract. The main goal is to comprehend how these operators can be utilized to fully describe and solve the problem of differential equations with boundary conditions for a second-order differential operator. These operators can also be applied to analyse and solve problems pertaining to graph structures, system formulation of a boundary value problems operator on graphs, and boundary value problems operator on graphs.

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## 1. INTRODUCTION

Introduction Let $\Gamma$ be a graph with finitely many edges, say $m$, each of finite lengths. Denote the edges by $E=\{e_j\}, j=1,2,3,...,m$, and the corresponding lengths of the edges by $L_j,\ j=1,2,3,...,m,$. as well as matching edge lengths by $l_j,\ j=1,2,3,...,m,$. Where $e$ is a real value function on $\Gamma$. Constraints are imposed on $p$ as because graphs require extra structure. Formally, these boundary conditions are known as self-adjoint boundary conditions, and we also understand $p$ faction [1].

The formal second-order differential operator is taken into consideration.

$$Ly(x)-\frac{d^2y(x)}{dx^2}+p(x)y(x)=\lambda y(x) \tag{1}$$

Let $\Gamma=\{V,E\}$ be a graph, where $V$ is the set of vertices, vertices 0, 1, …, $m$ are called the star graph's boundary vertices. numbered and $E$ is the set of edges $e_1,e_2,...,e_m$ of the graph $\Gamma=\{V,E\}$ [2]. On each edge $e_j$, the following $j$-th differential equation and in particular with equation (1):

$$-\frac{d^2y_j}{dx^2}+p_j(x)y_j=\lambda y_j, \quad x\in(0,l_j),\ j=1,2,3,...,k \tag{2}$$

where restricted to $e_j$, and $e_j$ is identified with $(0,l_j)$. The following boundary conditions are considered,

$$\alpha_j\left(y_j(0)+y_j(l_j)\right)+\beta_j\left(y'(0)+y'(l_j)\right)=0, \quad j=1,2,3,...,k \tag{3}$$

where the quantity of boundary conditions that are linearly independent is $k$. Although necessary, there are not enough linearly independent boundary conditions to ensure that the boundary value problem is self-adjoint on $\Gamma$. The Sturm-Liouville operator on a graph has self-adjoint boundary conditions that have been described by Harmer [2], Kostrykin [3], Schrader and Kuchment in [4]. Carlson [1] describes a class of differential operators on a weighted graph, including their adjoints and domains of fundamental self-adjointness. For more additional on boundary value problems' self-adjointness on a star-shaped graph.

Problem statement: Let's go over the outcomes of the Dirichlet problem solution for quadratic differential operators on lattice-type manifolds in more detail. The problem of differential equations with boundary conditions for a second-order differential operator on one is described and thoroughly solved using a hybrid approach in this study.

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The opinion framework is based on second-order differential equation in graphs with boundary conditions, eigenvalues and eigenfunction to obtain general solutions and private solutions in boundary conditions, boundary value problems of differential operators in graphs are differential operators, and widely used in economic sciences, other dimensions in society are used.

The remainder of this article is organized as follows. In Section 2, a brief review of some concepts and techniques for sorting differential equations on graphs is provided, along with boundary problem conditions and initial conditions. In section 3, the methodology is described. In section 4, the definition of the operator of boundary value problems is clearly stated on the graphs. In section 5, the formulation of the operator system of boundary value problems on graphs is stated with theorems and examples, and conclusions are drawn in section 6.

## 2. LITERATURE REVIEW

Over the past quarter-century to three decades, the field of differential equations and boundary value problems on geometric graphs, or spatial networks, has undergone a period of intensive development, as evidenced by the proliferation of scholarly works on the subject. This line of research originated in the seminal contributions of such mathematicians as B.S. Pavlov [5], Yu.V. Pokorny, and O.M. Penkin ([6], [7]), as well as the efforts of their international counterparts, including J. von Below ([8], [9]), G. Lumer [10], and S. Nicaise [11], along with the pioneering work of Dirichlet and his contemporaries [1-4,12-13]. These works have addressed a variety of models and phenomena, ranging from diffusion processes to the oscillation of elastic networks and the propagation of nervous impulse. The main results of B.E. Kanguzhina, L.K. Zhapsarbaeva and N.P. Bondarenko are reflected in [14], [15], and [12] (see also the bibliography provided in these works). In particular, in [15], the authors studied the Green's function for the Dirichlet problem on a graph star and presented relevant expansion theorems. Research objectives: We can see our objectives of the research explicitly mention that the research focuses on the boundary value problems associated with the differential equations on the star-shaped graphs.

## 3. METHODOLOGY

The research method is based on second-order linear differential equations, with boundary problem conditions and boundary problem differential operators on graphs and Green's function. The collection of information about the format of the library, website, domestic and foreign scientific articles, master's theses and doctoral theses has been compiled.

## 4. DEFINITION OF A BOUNDARY VALUE PROBLEMS OPERATOR ON GRAPHS

In what follows it is useful to introduce the space $L^2(G)$ on each edge $e_j$, the following $j$-th differential equation and in particular by equation (1) we mean the system of equations:

$$-y_j + p_j(x)y_j = \lambda y_j, \quad x \in (0, l_j),\ j = 1, 2, 3, ..., k \tag{4}$$

where $p_j$ and $y_j$ denote $p$ and $y$ restricted to $e_j$, and $e_j$ is identified with $(0, l_j)$.

The part where the star graphs in the second order differential equations were defined in the introduction section can be seen in relation (1), (2) and (3). According to the proposed work, For the second-order differential operator on a star graph, the properties of Green's functions of a boundary value issue have been examined.

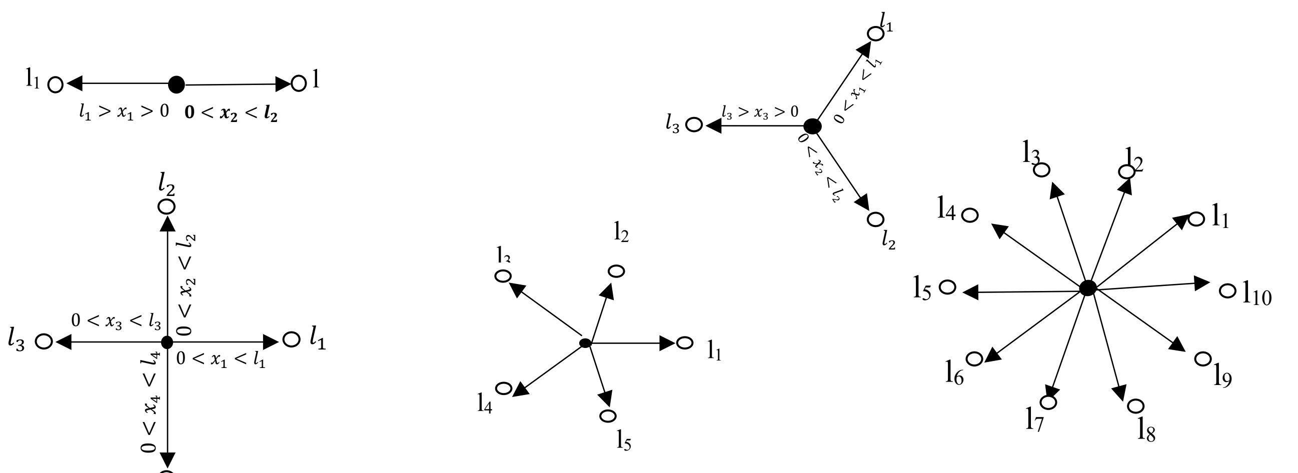

**Figure 1.** Examples of star-graph $m = 2, 3, 4, 5, 10$

An operator eigenvalue issue in $L^2(G)$, for the closed densely defined operator can be used to formulate the boundary value problems (1), (2), (3), and (4) on $G$ [2].

$$Ly = -f'' + pf \tag{5}$$

using with a domain

$$D(L) = \left\{ f \setminus f, f' \in AC,\ Lf \in L^2(G) \right\} \tag{6}$$

It should be known that $f$ following the rules of the equation (3).

## 5. SYSTEM FORMULATION OF A BOUNDARY VALUE PROBLEMS OPERATOR ON GRAPHS

A boundary value problem for a system on the interval $x \in (0, l_j)$ can be reformulated from the boundary value problem on a graph. Consider the edge $e_i$ of length $L_i$, we then have:

$$-y_j'' + p_j(x) y_j(x) = \lambda y_j(x), \quad x \in (0, l_j).$$

Let $t = \dfrac{x}{l_j}$ and $\tilde{y}_j(t) = y_j(l_j t).$ Then $-\dfrac{d^2}{dt^2}\left(\tilde{y}_j(t)\right) = -l_j^2\, y_j''(l_j t) = l_j^2 \left(\lambda y_j(l_j t) - p_j(l_j t) y_j(l_j t)\right) = l_j^2 \left(\lambda - P_j(t)\right) \tilde{y}_j(l_j t),$ where $P_j(t) = p_j(l_j t)$ Thus for each $j = 1, 2, 3, ..., k$ our transformed equation is

$$-\tilde{y}_j'' + l_j^2 (P_j - \lambda) \tilde{y}_j = 0, \quad x \in (0,1)$$

giving the system

$$\tilde{L}\tilde{Y} = -W\tilde{Y}'' + P\tilde{Y} = \lambda \tilde{Y} \tag{7}$$

where $W = diag\left[\dfrac{1}{l_1^2}, ..., \dfrac{1}{l_k^2}\right]$, $\tilde{Y} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_k \end{bmatrix}$ and $P = diag\left[p_1, ..., p_k\right]$. The boundary conditions are the next thing we look at. Since we only have the endpoints at 0 and 1 after completing the aforementioned transformation on each edge, the boundary conditions can be expressed in matrix form as follows:

$$\tilde{A}\tilde{Y}(0) + \tilde{B}\tilde{Y}'(0) + \tilde{C}\tilde{Y}(1) + \tilde{D}\tilde{Y}'(1) = 0 \tag{8}$$

where $\tilde{A} = \left[\alpha_j\right]$, $\tilde{B} = \left[\dfrac{\beta_j}{l_j}\right], \tilde{C} = \left[\beta_j\right]$, and $\tilde{D} = \left[\dfrac{\beta_j}{l_j}\right]$. The system boundary value issue with differential equation is therefore equal to the initial boundary value problem on the graph (7) and boundary conditions (8).

Let $L_k^2 = \left\{ F : (0,1) \to C^k \setminus F_i \in L^2(0,1),\ j = 1, 2, 3, ..., k \right\}$ utilizing the internal product

$$< F, G > W = \sum_{j=1}^{K} l_j \int_0^1 F_j \tilde{G}_j dt = \int_0^1 F^T W^{-\frac{1}{2}} \tilde{G} dt \tag{9}$$

It should be noted that $L_K^2$ is isometric ally isomorphic to $L^2(G)$ under the identification $L^2(G) \to L_k^2$ defined by

$$f(x) \to \begin{bmatrix} f/_{e_1}(l_1 t) \\ f/_{e_2}(l_2 t) \\ \vdots \\ f/_{e_k}(l_k t) \end{bmatrix}$$

where $x \in G$ and $t \in (0,1)$.
As an operator eigenvalue problem, the boundary value problems (7) and (8) can be rewritten with inner, [3,13], by setting $\tilde{L}F = -WF'' + PF$ using a domain

$$D(\tilde{L}) = \left\{ F \setminus F, F' \in AC, \ \ \tilde{L}F \in L^2(G), \ obeying\,(8) \right\}.$$

**Theorem.** The system (7) and (8) is formally self-adjoint if only the boundary value problem (2) and (3) in $L^2(G)$ is formally self-adjoint.

**Proof:** Let $F, G : (0,1) \to C^k$ be $C^2$ and denote by $f$ and $g$ the functions on $G$ defined by $f \setminus_{e_j}(l_j t) = F_j(t)$ and $g \setminus_{e_j}(l_j t) = G_j(t)$ for $j = 1,2,3,...,k$ and $t \in (0,1)$. Consequently, under this identification

$$-< G, \tilde{L}F > W = -\sum_{j=1}^{k} l_j^{-1} \int_0^1 \left[ F_j'' \tilde{G}_j - F_j \tilde{G}_j'' \right] dt = -\sum_{j=1}^{k} l_j^{-1} \left[ F_j' \tilde{G}_j - F_j \tilde{G}_j' \right]_{x=0}^{x=1} = \sum_{j=1}^{k} \left[ \left( f' \tilde{g} - f \tilde{g}' \right) |_{e_j} \right]_{x=0}^{x=l_j} = (Lf, g) - (f, Lg)$$

and (3) holds if and only if (8).

Given the circumstances, the formal self-ad jointness of (7) and (8) ensures that the operator $\tilde{L}$ on $L_k^2$ The formal self-adjointness of is a closed densely defined self-adjoint operator (2) and (3) ensures that $\tilde{L}$ is a closed densely defined self-adjoint operator in $L_k^2$.

**Example.** We demonstrate with a counter-example that even a straightforward self-adjoint boundary value problem on a graph does not require regularity. Consider the graph with one node, $L$ , and the second order operator

$$-\frac{d^2 y(x)}{dx^2} + p(x)y(x) = \lambda y(x),$$
$$y(0) = y(1),$$
$$y'(0) = y'(1),$$

at $L$. After that, we have that:

$$\Delta = -\det \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = 0$$

i.e. we don't have regularity. As demonstrated by the example above, most self-adjoint problems on graphs are not regular.

The theorem has applications in mathematics, particularly in the analysis of ordinary differential equations, and the systems of equations associated with them are generally commutative and, in terms of differential equation theory, provide a condition under which solutions to the boundary value problem exist uniquely. This theorem can be used to simplify the analysis of complex systems.

## 6. CONCLUSIONS

In this paper attempts to formulate the operator of boundary value problems on graphs and system formulation of a second-order differential operator's boundary value problems operator on graphs of differential equations. Using an example and theory in the theorem have also helped to clarify the results.

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## DECLARATION OF ORIGINALITY

The authors declare no conflict of interest to report regarding this study conducted.

## REFERENCES

[1] Carlson R. Adjoint and self-adjoint differential operators on graphs: Electronic Journal of Differential Equations 1998; 6:1-10.

[2] Kanguzhin B, Aimal Rasa GH, Kaiyrbek Z. Identification of the domain of the Sturm–Liouville operator on a star graph. Symmetry. 2021 Jul 6;13(7):1210.

[3] Rasa GH. Residual Decomposition of the Green's Function of the Dirichlet problem for a Differential Operator on a star-graph for m= 2. Turkish Journal of Computer and Mathematics Education. 2023 Apr 21;14(2):132-40.

[4] Rasa GH, Auzerkhan GS, Konyrkulzhayeva MN. Green's function for the Dirichlet problem of a differential operator on a star graph with m. Journal of Mathematics, Mechanics and Computer Science. 2019 Apr 23;101(1):14-28.

[5] Shirshin EA, Tikhonova TN, Budylin GS, Fadeev VV. Detection of metal nanoparticles in aqueous solutions using laser-induced optical breakdown. Processes in Geoenvironmental. 2016; 4:371-7.

[6] Pokorny YuV. On the spectrum of some problems on graphs. Uspekhi Mat.Nauki. 1987; 42(4):128-9.

[7] Penkin OM. On a boundary value problem on a graph. Journal of Differential Equations. 1988; 24(4):701-3.

[8] Von Below J. Classical solvability of linear parabolic equations on networks. Journal of Differential Equations. 1988; 72(2):316-7.

[9] Von Below J. Sturm-Liouville eigenvalue problems on networks. Mathematical Methods in the Applied Sciences. 1988;10(4):383-5.

[10] Lumer G. Connecting of local operators and evolution equations on networks. InPotential Theory Copenhagen 1979: Proceedings of a Colloquium Held in Copenhagen, May 14–18, 1979 1980 (pp. 219-234). Springer Berlin Heidelberg.

[11] Nicaise S. Some results on spectral theory over networks, applied to nerve impulse transmission. InPolynômes Orthogonaux et Applications: Proceedings of the Laguerre Symposium held at Bar-le-Duc, October 15–18, 1984 1985 (pp. 532-541). Springer Berlin Heidelberg.

[12] Bondarenko NP. A 2-edge partial inverse problem for the Sturm-Liouville operators with singular potentials on a star-shaped graph. arXiv preprint arXiv:1702.08293. 2017 Feb 27.

[13] Rasa GR, Rasa GH. Sturm-Liouville problem with general inverse symmetric potential. Science and Education. 2023;4(8):7-15.

[14] Глотов НВ. Differential equations on geometric graphs with singularities in coefficients. PhD Thesis. Russia: University Voronezh, 2007.

[15] Kanguzhin BE. Green's function for the Dirichlet problem of a differential operator on a star graph. Journal of Mathematics, Mechanics and Computer Science. 2018 Aug 27;97(1):67-90.