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RESEARCH ARTICLE

Finite cyclic group of *p*-power order and its compatibility conditions

F.H. Hasan[1,2], M.S. Mohamad[2]*, Y. Yusof[2] and N.A. Mohd Busul Aklan[3]

[1]Tunku Abdul Rahman University of Management and Technology, Pahang Branch, Bandar Indera Mahkota, 25200 Kuantan, Pahang, Malaysia

[2]Centre for Mathematical Sciences, Universiti Malaysia Pahang Al-Sultan Abdullah, Lebuh Persiaran Tun Khalil Yaakob, 26300 Kuantan, Pahang, Malaysia

[3]Department of Computational and Theoretical Sciences, Kulliyah of Science, International Islamic University Malaysia, 25200 Kuantan, Pahang, Malaysia

ABSTRACT - Finite cyclic groups of *p*-power order, where *p* represents a prime number, have long been an interesting field of study in abstract algebra. This paper investigates the compatibility conditions that control their existence and behaviour. An overview of cyclic groups, automorphisms and their properties is given as groundwork for this research. By analysing the interaction between the group's order and its generator, we discovered the compatibility conditions and presented them as the primary finding in this paper.

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1. INTRODUCTION

Brown and Loday revolutionized the world of mathematics with their groundbreaking approach of the nonabelian tensor product of groups in 1984 [1]. In 1987, Brown et al. utilized the generic Van Kampen Theorem to determine the source of the non-abelian tensor product [2]. The structure of the nonabelian tensor product was initially discovered in homotopy theory and algebraic K-theory. This discovery has since contributed significantly to our understanding of the subject matter and has opened new avenues for research in mathematics. Its impact continues to be recognized and studied by experts in the field. According to Brown et al. [2] the nonabelian tensor product $G \otimes H$ is a group producing symbols $g \otimes h$ if G and H are groups that act compatibly with one another, illustrated as follows:

$$^{(^g h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } {}^{(^h g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

where $g, g' \in G$ and $h, h' \in H$ [2].

In 1997, Kappe gave an overview and literature review on nonabelian tensor product [3]. Brown et al. extended their previous research by delving into the group theoretical features of the nonabelian tensor product. Specifically, they explored the computation of the nonabelian tensor square [2]. Additionally, they offered fellow academics an opportunity to investigate and analyse the group theoretical characteristics associated with the nonabelian tensor product of groups. To facilitate this exploration, they presented a compilation of open topics linked to both the nonabelian tensor product and nonabelian tensor square. A portion of the unsolved issues regarding the nonabelian tensor product for cyclic groups motivated this study.

The automorphism group must be established before the nonabelian tensor product can be calculated. Since this research primarily focuses on the cyclic group, it is important to note that a group, *G*, can only be considered cyclic if a single element or group generator forms it. Considering that a finitely generated group is a central extension of a cyclic group and is finite, in 1962, Alperin discovered that it possesses a finite automorphism group [4]. He also provided a more straightforward proof of the theorem made by Baer [5], where processing only finitely many automorphisms, a group is considered finite [4]. Ahmad (1969) found the cycle structure of the group of automorphism of an arbitrary finite cyclic group that has a specific order, which is determined by a natural number, *n*. [6].

Dummit and Foote [7] illustrated the automorphism group of finite cyclic groups of *p*-power order, which derived directly as $\mathrm{Aut}(G) \cong C_{p-1} \times C_{p^{\alpha-1}} \cong C_{(p-1)p^{\alpha-1}}$, where *p* is an odd prime number and $\alpha \in \mathbb{N}$. Moreover, they proved that the automorphism of the *p*-power order is isomorphic to the explicit multiplication of two finite cyclic groups. This means that to find the order of automorphisms that have *p*-power order, one must first find the generator of the finite cyclic group. A study by Shahoodh in 2018 focused on the subsequent component of the explicit multiplication of the automorphism group of *p*-power order, which is $C_{p^{\alpha-1}}$. He determined the quantity of the automorphisms and the total count of automorphisms possessing the *p*-power order for each *p*-power order finite cyclic group, where *p* represents an

*CORRESPONDING AUTHOR | M.S. Mohamad | ✉ mohdsham@umpsa.edu.my

odd prime. Moreover, it was found that there is precisely an element of order two for every finite cyclic automorphism group of the *p*-power order [8].

Sommer-Simpson computed the order of $\mathrm{Aut}\left(\mathbb{Z}_n \times \mathbb{Z}_2\right)$ as well as the order of $\mathrm{Aut}\left(\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p\right)$, where *p* is prime. He provided general results for the structures of automorphism groups for semidirect products, specifically for the case $\mathrm{Aut}\left(K \rtimes H\right)$, where *H* and *K* are cyclic groups. Moreover, he included the general form for the automorphisms and automorphism groups of dihedral groups. Lastly, he computed the automorphism of the semidirect product of $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$ [9]. Emery discussed the automorphism groups of finite groups of diminutive order and listed some general results on the automorphism of the cyclic group [10].

To determine a nonabelian tensor product, it is necessary to ensure that two groups interact with one another and satisfy the compatibility conditions. This compatibility step guarantees a well-defined and meaningful calculation of the nonabelian tensor product. Visscher [11] established several essential requirements for two cyclic groups to interact compatibly. However, these criteria only apply when both actions have order two and are one-sided. Additionally, Visscher pointed out that the cyclic groups are compatible when either both or one of the actions is trivial. Mohamad [12] established a novel correlation between the order of actions and compatibility, specifically for definite cyclic 2-groups and definite cyclic *p*-groups where *p* represents an odd prime number.

The accurate number of compatible pairings of actions between a pair of 2-power order cyclic groups as well as results of the compatible pairings of nontrivial actions of orders two and four were provided by Sulaiman et al. [13]. Sulaiman et al. specifically addressed the case where one of the actions has an order higher than two and demonstrated the precise number of compatible pairings of actions for the finite cyclic groups of 2-power order [14]. In 2017, Sulaiman et al. established compatible pairings of nontrivial actions within finite cyclic groups of 2-power order and their subgroup [15].

The compatible pairs of 3-power order finite cyclic groups were found by Shahoodh et al. in 2016 [16]. Mohamad et al. precisely found the number of compatible pairings of nontrivial actions for identical cyclic groups of 2-power order with actions of similar order [17]. Shahoodh et al. provided necessary criteria for two actions to be compatible and have *p*-power, where *p* represents an odd prime. He also calculated the precise number of actions from the same group that were compatible with one another [8]. Ultimately, he only covered the subsequent half of the direct amalgamation of the *p*-power order of the automorphism group.

The paper aims to present the compatibility conditions for finite cyclic groups of order *p*, where *p* is an odd prime. Although the conditions for finite cyclic groups of 2-power and *p*-power order have already been discussed separately, this paper provides an overview of the compatibility conditions for all finite cyclic groups of order *p* as a whole. With the aid of the Groups, Algorithm, and Programming (GAP) Software, the compatibility criteria for the finite cyclic groups of *p*-power order where *p* represents an odd prime number are identified [18].

2. METHODOLOGY

First, the automorphism of the finite cyclic group of *p*-power order can be illustrated as a direct multiplication of a pair of finite cyclic groups, as shown in the subsequent theorem.

Theorem 2.1 [7]

Given that p represents an odd prime and $\alpha \in \mathrm{N}$. Assume G is a cyclic group of order p^{α}, $\mathrm{Aut}(G) \cong C_{p-1} \times C_{p^{\alpha-1}} \cong C_{(p-1)p^{\alpha-1}}$ and $\left|\mathrm{Aut}(G)\right| = (p-1)p^{\alpha-1}$.

The subsequent theorem explains the order of any given integer, *a*.

Theorem 2.2 [19]

Given that an integer, *a* modulo *n* with order *k* and integer $h > 0$, then a^h has order $\left(\dfrac{k}{\gcd(h,k)}\right)$ modulo *n*.

The following theorem shows the classification of the generator.

Theorem 2.3 [20]

Let $n \geq 2$, then U_n is having an element *a* fulfilling $U_n = \langle a \rangle$ provided that *a* takes any of the subsequent structures: 2, 4, p^k, or $2p^k$ where *p* represents an odd prime and $k \in \mathbb{N}$.

Note that, by letting $C_{p^{\alpha}} = \langle g \rangle$ and $\phi(g) = g^n$ is an automorphism of $C_{p^{\alpha}}$, then $(n, p^{\alpha}) = 1$. This is equivalent to the definition of a Unitary group. According to Theorem 2.1, if $\mathrm{Aut}(C_{p^{\alpha}})$ is a cyclic group, then $\mathrm{Aut}(C_{p^{\alpha}}) \cong U_{p^{\alpha}}$.

Next, the definition of the group's action is presented.

Definition 2.1 [2]

An action of group, G on another group, H is defined as a mapping $\Phi: G \rightarrow \mathrm{End}(H)$ where $\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$ for all $g, g' \in G$ and $h \in H$.

Given that G and H are both finite cyclic groups, the action Φ of group G on group H must consider which is the identity mapping on group H. Hence, the action is homomorphism from group G to the $\mathrm{Aut}(H)$.

The compatible pairings of actions between two groups are described as follows.

Definition 2.2 [2]

Assuming the interaction between groups, G and H occurs independently through conjugation. The next step involves ensuring that the actions taken are mutually compatible as follows.

$$^{(^g h)}g' = {}^g({}^h({}^{g^{-1}}g')) \tag{1}$$

and

$$^{(^h g)}h' = {}^h({}^g({}^{h^{-1}}h')) \tag{2}$$

where $g, g' \in G$ and $h, h' \in H$.

Next, the proposition on the compatibility conditions for abelian groups is given.

Proposition 2.1 [11]

Let G and H represent the groups that act with one another. When G and H, which are both abelian groups, act on each other, their actions are said to be compatible if and only if $^{(^g h)}g' = {}^h g'$ and $^{(^h g)}h' = {}^g h'$ for all $g, g' \in G$ and $h, h' \in H$.

The subsequent theorem outlines the necessary and sufficient condition ensuring compatibility between two cyclic groups of 2-power.

Theorem 2.4 [12]

Let $G = \langle x \rangle = C_{2^m}$ and $H = \langle y \rangle = C_{2^n}$ with the actions of y on x and vice versa given by such that ${}^y x = x^s$ and ${}^x y = y^t$ where s and t are positive odd integers. Assume the actions of y on x has order 2^k and the action of x on y has order 2^l. If and only if $s \equiv 1 \bmod 2^k$ and $t \equiv 1 \bmod 2^l$, then the actions are considered as compatible.

3. RESULTS AND DISCUSSION

If $n \geq 2$, then $\mathrm{Aut}(C_n)$ consists of an element x that satisfies $\mathrm{Aut}(C_n) = \langle x \rangle$ and provided that x has any of the subsequent structure: 2, 4, p^r, or $2p^r$, where p represents an odd prime and $r \in \mathbb{N}$. Let G acts on H in such a manner that ${}^g h$ and H acts on G in such a manner that ${}^h g$ as well as both generators are the same. For an action to be considered compatible, it must meet at least one of the following criteria.

Table 1. Conditions for p-power order cyclic groups' compatibility

Generator	${}^h g$	${}^g h$	Compatibility conditions
2	g^{2^k}	h^{2^l}	$2^k(2^{2^l}-1) \equiv 0 \bmod \lvert Aut(G)\rvert$ and $2^l(2^{2^k}-1) \equiv 0 \bmod \lvert Aut(H)\rvert$
4	g^{4^k}	h^{4^l}	$2^{2k}(2^{2^{2l}}-1) \equiv 0 \bmod \lvert Aut(G)\rvert$ and $2^{2l}(2^{2^{2k}}-1) \equiv 0 \bmod \lvert Aut(H)\rvert$
p^r	$g^{p^{rk}}$	$h^{p^{rl}}$	$p^{rk}(p^{p^{rl}}-1) \equiv 0 \bmod \lvert Aut(G)\rvert$ and $p^{rl}(p^{p^{rk}}-1) \equiv 0 \bmod \lvert Aut(H)\rvert$
$2p^r$	$g^{2p^{rk}}$	$h^{2p^{rl}}$	$p^{rk}((2p)^{2p^{rl}}-2) \equiv 0 \bmod \lvert Aut(G)\rvert$ and $p^{rl}((2p)^{2p^{rk}}-2) \equiv 0 \bmod \lvert Aut(H)\rvert$

To prove the above statement, we consider several cases given as Lemma 3.1 to 3.4.

Lemma 3.1

Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be cyclic groups of prime power order with $\mathrm{Aut}(C_{p^\alpha}) = \langle 2 \rangle$ and $\mathrm{Aut}(C_{p^\beta}) = \langle 2 \rangle$, where g is an element belongs to G and h is an element belongs to H and $\alpha, \beta \in \mathbb{N}$. Let G acts on H in a way that ${}^g h = h^{2^l}$ and H acts on G such that ${}^h g = g^{2^k}$. Subsequently, the actions act compatibility if $2^k(2^{2^l}-1) \equiv 0 \bmod \left|\mathrm{Aut}(C_{p^\alpha})\right|$ and $2^l(2^{2^k}-1) \equiv 0 \bmod \left|\mathrm{Aut}(C_{p^\beta})\right|$, where $k, l \in \mathbb{N}$.

Proof: If $G=\langle g\rangle \cong C_{p^{\alpha}}$ and $H=\langle h\rangle \cong C_{p^{\beta}}$ be cyclic groups of prime power order with $\mathrm{Aut}(C_{p^{\alpha}})=\langle 2\rangle$ and $\mathrm{Aut}(C_{p^{\beta}})=\langle 2\rangle$, then G acts on H such that ${}^{g}h=h^{2^{l}}$ and H acts on G such that ${}^{h}g=g^{2^{k}}$. With reference to Theorem 2.1, $\left|\mathrm{Aut}(C_{p^{\alpha}})\right|=(p-1)p^{\alpha-1}$. According to Proposition 2.1, it needs to meet the compatibility requirements such that ${}^{({}^{g}h)}g'={}^{h}g'$ and ${}^{({}^{h}g)}h'={}^{g}h'$, where $g,g'\in G$ and $h,h'\in H$. By considering the primary condition, hence,

$$
\begin{aligned}
{}^{(h^{2^{l}})}g' &= g'^{(2^{k})}\\
g'^{(2^{k})^{2^{l}}} &= g'^{(2^{k})}
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
(2^{k})^{2^{l}} &\equiv 2^{k} \bmod \left|\mathrm{Aut}(C_{p^{\alpha}})\right|\\
(2^{k})^{2^{l}} &\equiv 2^{k} \bmod (p-1)(p^{\alpha-1})\\
2^{k\cdot 2^{l}}-2^{k} &\equiv 0 \bmod (p-1)(p^{\alpha-1})\\
2^{k}(2^{2^{l}}-1) &\equiv 0 \bmod (p-1)(p^{\alpha-1})
\end{aligned}
$$

Thus, the first compatibility condition is satisfied. Similar proving can be applied to the second condition. □

Lemma 3.2

Let $G=\langle g\rangle \cong C_{p^{\alpha}}$ and $H=\langle h\rangle \cong C_{p^{\beta}}$ be cyclic groups of prime power order, with $\mathrm{Aut}(C_{p^{\alpha}})=\langle 4\rangle$ and $\mathrm{Aut}(C_{p^{\beta}})=\langle 4\rangle$, where g is an element belongs to G and h is an element belongs to H and $\alpha,\beta\in$ □ . Let G acts on H such that ${}^{g}h=h^{4^{l}}$ and H acts on G such that ${}^{h}g=g^{4^{k}}$. Then, the actions act compatibility if $2^{2k}(2^{2^{2l}}-1)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^{\alpha}})\right|$ and $2^{2l}(2^{2^{2k}}-1)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^{\beta}})\right|$, where $k,l\in$ □ .

Proof: If $G=\langle g\rangle \cong C_{p^{\alpha}}$ and $H=\langle h\rangle \cong C_{p^{\beta}}$ be cyclic groups of prime power order, with $\mathrm{Aut}(C_{p^{\alpha}})=\langle 4\rangle$ and $\mathrm{Aut}(C_{p^{\beta}})=\langle 4\rangle$. Let G acts on H such that ${}^{g}h=h^{4^{l}}$ and H acts on G such that ${}^{h}g=g^{4^{k}}$. According to Theorem 2.1, $\left|\mathrm{Aut}(C_{p^{\alpha}})\right|=(p-1)p^{\alpha-1}$. By Proposition 2.1, it has to satisfy the compatibility conditions such that ${}^{({}^{g}h)}g'={}^{h}g'$ and ${}^{({}^{h}g)}h'={}^{g}h'$, where $g,g'\in G$ and $h,h'\in H$. Now, considering the first condition in such a manner that,

$$
\begin{aligned}
{}^{h^{(2^{2l})}}g' &= g'^{(2^{2k})}\\
g'^{(2^{2k})^{2^{2l}}} &= g'^{(2^{2k})}
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
(4^{k})^{4^{l}} &\equiv 4^{k} \bmod \left|\mathrm{Aut}(C_{p^{\alpha}})\right|\\
(4^{k})^{4^{l}} &\equiv 4^{k} \bmod (p-1)(p^{\alpha-1})\\
(2^{2k})^{2^{2l}} &\equiv 2^{2k} \bmod (p-1)(p^{\alpha-1})\\
2^{2k\cdot 2^{2l}} &\equiv 2^{2k} \bmod (p-1)(p^{\alpha-1})\\
2^{2k\cdot 2^{2l}}-2^{2k} &\equiv 0 \bmod (p-1)(p^{\alpha-1})\\
2^{2k}(2^{2^{2l}}-1) &\equiv 0 \bmod (p-1)(p^{\alpha-1})
\end{aligned}
$$

Therefore, the first compatibility condition is satisfied. Likewise, it implies the second condition. □

Lemma 3.3

Let $G=\langle g\rangle \cong C_{p^{\alpha}}$ and $H=\langle h\rangle \cong C_{p^{\beta}}$ be cyclic groups of prime power order, with $\mathrm{Aut}(C_{p^{\alpha}})=\langle p^{r}\rangle$ and $\mathrm{Aut}(C_{p^{\beta}})=\langle p^{r}\rangle$, where g is an element belongs to G and h is an element belongs to H and $\alpha,\beta\in$ □ . Then, G acts on H in such a manner that ${}^{g}h=h^{p^{rl}}$ and H acts on G such that ${}^{h}g=g^{p^{rk}}$. Then, the actions act compatibility if $p^{rk}(p^{p^{rl}}-1)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^{\alpha}})\right|$ and $p^{rl}(p^{p^{rk}}-1)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^{\beta}})\right|$, where $k,l,r\in$ □ .

Proof: $G=\langle g\rangle \cong C_{p^\alpha}$ and $H=\langle h\rangle \cong C_{p^\beta}$ be cyclic groups of prime power order, with $\text{Aut}(C_{p^\alpha})=\langle p^r\rangle$ and $\text{Aut}(C_{p^\beta})=\langle p^r\rangle$, then G acts on H such that ${}^g h = h^{p^{rl}}$ and H acts on G such that ${}^h g = g^{p^{rk}}$. According to Theorem 2.1, $\left|\text{Aut}(C_{p^\alpha})\right| = (p-1)p^{\alpha-1}$. By Proposition 2.1, it has to satisfy the compatibility conditions such that ${}^{({}^g h)}g' = {}^h g'$ and ${}^{({}^h g)}h' = {}^g h'$, where $g,g'\in G$ and $h,h'\in H$. Now, considering the first condition in such a manner,

$$^{(h^{p^{rl}})}g' = g'^{(p^{rk})}$$
$$g'^{(p^{rk})^{p^{rl}}} = g'^{(p^{rk})}$$

or equivalently,

$$(p^{rk})^{p^{rl}} \equiv p^{rk} \bmod \left|\text{Aut}(C_{p^\alpha})\right|$$
$$(p^{rk})^{p^{rl}} \equiv p^{rk} \bmod (p-1)(p^{\alpha-1})$$
$$p^{rk\cdot p^{rl}} \equiv p^{rk} \bmod (p-1)(p^{\alpha-1})$$
$$p^{rk\cdot p^{rl}} - p^{rk} \equiv 0 \bmod (p-1)(p^{\alpha-1})$$
$$p^{rk}(p^{p^{rl}}-1) \equiv 0 \bmod (p-1)(p^{\alpha-1})$$

Thus, the first compatibility criteria are satisfied. Similar proving can be applied to the second case. □

Lemma 3.4

Let $G=\langle g\rangle \cong C_{p^\alpha}$ and $H=\langle h\rangle \cong C_{p^\beta}$ be cyclic groups of prime power order, with $\text{Aut}(C_{p^\alpha})=\langle 2p^r\rangle$ and $\text{Aut}(C_{p^\beta})=\langle 2p^r\rangle$, where g is an element belongs to G and h is an element belongs H and $\alpha,\beta\in\mathbb{N}$. Then, G acts on H such that ${}^g h = h^{2p^{rl}}$ and H acts on G such that ${}^h g = g^{2p^{rk}}$. Hence, the action is compatible if $p^{rk}((2p)^{2p^{rl}}-2)\equiv 0 \bmod \left|\text{Aut}(C_{p^\alpha})\right|$ and $p^{rl}((2p)^{2p^{rk}}-2)\equiv 0 \bmod \left|\text{Aut}(C_{p^\beta})\right|$, where $k,l,r\in\mathbb{N}$.

Proof: If $G=\langle g\rangle \cong C_{p^\alpha}$ and $H=\langle h\rangle \cong C_{p^\beta}$ be cyclic groups of prime power order, with $\text{Aut}(C_{p^\alpha})=\langle 2p^r\rangle$ and $\text{Aut}(C_{p^\beta})=\langle 2p^r\rangle$, then G acts on H such that ${}^g h = h^{2p^{rl}}$ and H acts on G such that ${}^h g = g^{2p^{rk}}$. According to Theorem 2.1, $\left|\text{Aut}(C_{p^\alpha})\right| = (p-1)p^{\alpha-1}$. By Proposition 2.1, it has to satisfy the compatibility conditions such that ${}^{({}^g h)}g' = {}^h g'$ and ${}^{({}^h g)}h' = {}^g h'$, where $g,g'\in G$ and $h,h'\in H$. Now, considering the first requirement in such a manner,

$$^{(h^{2p^{rl}})}g' = g'^{(2p^{rk})}$$
$$g'^{(2p^{rk})^{2p^{rl}}} = g'^{(2p^{rk})}$$

or equivalently,

$$(2p^{rk})^{2p^{rl}} \equiv 2p^{rk} \bmod \left|\text{Aut}(C_{p^\alpha})\right|$$
$$(2p^{rk})^{2p^{rl}} \equiv 2p^{rk} \bmod (p-1)(p^{\alpha-1})$$
$$(2p^{rk})^{2p^{rl}} - 2p^{rk} \equiv 0 \bmod (p-1)(p^{\alpha-1})$$
$$2^{2p^{rl}}\, p^{rk\cdot 2p^{rl}} - 2p^{rk} \equiv 0 \bmod (p-1)(p^{\alpha-1})$$
$$p^{rk}(2^{2p^{rl}}\, p^{2p^{rl}} - 2) \equiv 0 \bmod (p-1)(p^{\alpha-1})$$
$$p^{rk}((2p)^{2p^{rl}} - 2) \equiv 0 \bmod (p-1)(p^{\alpha-1})$$

Therefore, the first compatibility criteria are satisfied. Likewise, it implies the second case. □

Theorem 3.1

Let $G=\langle g\rangle \cong C_{p^\alpha}$ and $H=\langle h\rangle \cong C_{p^\beta}$ be cyclic groups of prime power order where g is an element in G and h is an element in H and $\alpha,\beta\in\mathbb{N}$. Hence, g and h have one of the subsequent structures: 2, 4, p^r, or $2p^r$ where p is an odd prime and $r\in\mathbb{N}$. Moreover, let G acts on H such that ${}^g h$ and H acts on G such that ${}^h g$ as well as both generators are the same. Therefore, the action is considered compatible if it satisfies any of the following criteria as shown in Table 1.

Proof: Given that $G=\langle g\rangle \cong \mathrm{Aut}(C_{p^\alpha})$ and $H=\langle h\rangle \cong \mathrm{Aut}\left(C_{p^\beta}\right)$ be cyclic groups of prime power order, then, *G* acts on *H* such that ${}^{g}h$ and *H* acts on *G* such that ${}^{h}g$. Proposition 2.1 states that for something to be valid, it must meet certain compatibility conditions such that ${}^{({}^{g}h)}g' = {}^{h}g'$ and ${}^{({}^{h}g)}h' = {}^{g}h'$ where $g,g'\in G$ and $h,h'\in H$. As a result, we will consider the four cases listed as follows:

Case 1 (Generator 2): Suppose $G=\langle g\rangle \cong C_{p^\alpha}$ and $\mathrm{Aut}(C_{p^\alpha})=\langle 2\rangle$ along with $H=\langle h\rangle \cong C_{p^\beta}$ and $\mathrm{Aut}(C_{p^\beta})=\langle 2\rangle$. Then, by Lemma 3.1, the actions act compatibly if $2^k(2^{2^l}-1)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^\alpha})\right|$ and $2^l(2^{2^k}-1)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^\beta})\right|$.

Case 2 (Generator 4): Suppose $G=\langle g\rangle \cong C_{p^\alpha}$ and $\mathrm{Aut}(C_{p^\alpha})=\langle 4\rangle$ along with $H=\langle h\rangle \cong C_{p^\beta}$ and $\mathrm{Aut}(C_{p^\beta})=\langle 4\rangle$. By referring to Lemma 3.2, the actions act compatibly if $2^{2k}(2^{2^{2l}}-1)\equiv 0 \bmod \left|\mathrm{Aut}(\mathrm{C}_{p^\alpha})\right|$ and $2^{2l}(2^{2^{2k}}-1)\equiv 0 \bmod \left|\mathrm{Aut}(\mathrm{C}_{p^\beta})\right|$.

Case 3 (Generator p^r): Suppose $G=\langle g\rangle \cong C_{p^\alpha}$ and $\mathrm{Aut}(C_{p^\alpha})=\langle p^r\rangle$ along with $H=\langle h\rangle \cong C_{p^\beta}$ and $\mathrm{Aut}(C_{p^\beta})=\langle p^r\rangle$. Then, by Lemma 3.3, the actions act compatibly if $p^{rk}(p^{p^{rl}}-1)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^\alpha})\right|$ and $p^{rl}(p^{p^{rk}}-1)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^\beta})\right|$.

Case 4 (Generator $2p^r$): Suppose $G=\langle g\rangle \cong C_{p^\alpha}$ and $\mathrm{Aut}(C_{p^\alpha})=\langle 2p^r\rangle$ along with $H=\langle h\rangle \cong C_{p^\beta}$ and $\mathrm{Aut}(C_{p^\beta})=\langle 2p^r\rangle$. With reference to Lemma 3.4, the actions act compatibly if $p^{rk}((2p)^{2p^{rl}}-2)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^\alpha})\right|$ and $p^{rl}((2p)^{2p^{rk}}-2)\equiv 0 \bmod \left|\mathrm{Aut}(C_{p^\beta})\right|$.

Hence, the action is compatible if it satisfies one of the cases given in the theorem. □

4. CONCLUSIONS

This study provides a thorough analysis of the compatibility requirements for finite cyclic groups of order *p*, where *p* represents an odd prime, based on their generator. By examining their characteristics and interactions, we have gained a better understanding of the constraints that govern their compatibility. The study demonstrates the significance of the generator in determining the compatibility requirements for these groups. These findings expand our knowledge of finite cyclic groups and pave the way for further research into their structures and practical applications across various fields.

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NA

DECLARATION OF ORIGINALITY

The authors declare no conflict of interest to report regarding this study conducted.

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